CALCULUS FOR SCHOOLS

PART II

FAWDRY AND DURELL

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PART II

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PREFACE

THERE have been many changes in the teaching of elementary mathematics during the last twenty years, but none perhaps so significant as the growing tendency to regard any mathematical education as incomplete which does not include something of the fundamental ideas of the Calculus. This is a natural consequence of the belief that pupils towards the end of their time at School gain more from encountering new ideas than strengthening their power of technique in formal Algebra and Trigonometry. The authors have attempted in this volume to develop a systematic course which, while making small demands on the manipulative skill of the reader, calls for continual thought and makes progress depend rather on the exercise of common-sense than on the mechanical use of rules.

Few who have taught this subject to non-specialists will not agree that the real initial difficulty is of the same nature as that which presents itself in approaching Analytical Geometry. From the plotting of a graph from a given equation to a real grasp of the meaning of "the equation of a curve" is a big step, and it smooths the path to spend time in driving home this idea before embarking on the ordinary graphical illustrations which lead up to the peculiar notation of the Calculus. For this reason the authors attach great importance to the subject matter of Chapter I which might at a first glance perhaps appear irrelevant.

The discussions in the text have been cut down to the smallest dimensions consistent with indicating the plan of procedure and establishing the necessary bookwork. The nature of the course depends much more on the character of the exercises than on the actual bookwork and so every effort has been made to select

examples which exhibit the practical applications of the subject and the variety of its ideas.

The only discussion of any length in Part II is that which introduces the logarithmic and exponential functions. The method adopted is designed to show the reader that, just as the handling of the trigonometric functions is simplified by the change of measurement from degrees to radians by the introduction of π, so the inconvenience of working with an awkward numerical factor which appears in the differentiation of 10^x and $\log_{10} x$ is avoided by changing from the base 10 to a new base e whose value may be obtained approximately by using Simpson's rule. This emphasis on practical convenience tends to remove the mystery which so often surrounds e and e^x in the pupil's mind. A table of Napierian logarithms is provided to facilitate calculation and make the work more concrete.

Hyperbolic functions are slowly coming into general use: they are no more difficult to handle than the trigonometric functions, and offer the best means of evaluating various types of integrals. On account of their practical importance, a chapter is devoted to them and a table of values of the function for use in examples is given at the end of the book.

The scope of the whole work is indicated by the Table of Contents but it may be useful to mention that Parts I and II together include all that is required for the Army Entrance examination and the Qualifying examination for the Mechanical Sciences Tripos at Cambridge.

Acknowledgment is due to the Controller of H.M. Stationery Office and to the Syndics of the Cambridge University Press and to the Oxford and Cambridge Joint Board for kind permission to include questions set in recent examinations.

C. V. D.
R. C. F.

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PART II

CHAPTER XI

GENERAL METHODS OF DIFFERENTIATION

I. If u is any function of x and if C is a constant,

$$\mathbf{\frac{d}{dx}(Cu) = C\frac{du}{dx}.}$$

When any increment δx is made in x, the corresponding increment in u is denoted by δu.

$\therefore$ if $y = Cu$, $\quad y + \delta y = C(u + \delta u)$,

$$\therefore \delta y = C(u + \delta u) - Cu = C \,.\, \delta u,$$

$$\therefore \frac{\delta y}{\delta x} = C\frac{\delta u}{\delta x}.$$

$\therefore$ in the limit when $\delta x \to 0$

$$\frac{d}{dx}(Cu) = \frac{dy}{dx} = C\frac{du}{dx}.$$

Example 1.

$$\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

II. If u, v, w are any functions of x,

$$\mathbf{\frac{d}{dx}(u + v + w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots .}$$

Let $y = u + v + w + \ldots$ and when any increment δx is made in x, let the corresponding increments in $u, v, w \ldots y$ be denoted by $\delta u, \delta v, \delta w \ldots \delta y$.

Then $$y + \delta y = (u + \delta u) + (v + \delta v) + \ldots,$$

$$\therefore \delta y = \delta u + \delta v + \delta w + \ldots,$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x} + \ldots .$$

$\therefore$ in the limit when $\delta x \to 0$

$$\frac{d}{dx}(u+v+w+\ldots)=\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots.$$

Note that in the same way we may show that

$$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}.$$

Example 2.

Find $\frac{dy}{dx}$ when $y=3x^2-5x+4$.

$$\frac{dy}{dx}=\frac{d(3x^2)}{dx}-\frac{d(5x)}{dx}+\frac{d(4)}{dx}=6x-5.$$

Note, if u, x, y, z are all functions of t and $u=x+y+z$,

$$\frac{du}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}.$$

III. *Differentiation of a Product.*

If u, v are any functions of x,

$$\mathbf{\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}}.$$

Let $y=uv$, then with the same notation as in II. we have

$$y+\delta y=(u+\delta u)(v+\delta v),$$

$$\therefore\ \delta y=(u+\delta u)(v+\delta v)-uv$$

$$=u\,.\,\delta v+v\,.\,\delta u+\delta u\,.\,\delta v.$$

$$\therefore\ \frac{\delta y}{\delta x}=u\,.\,\frac{\delta v}{\delta x}+v\,.\,\frac{\delta u}{\delta x}+\frac{\delta u}{\delta x}\,.\,\frac{\delta v}{\delta x}\,.\,\delta x.$$

When $\delta x \to 0$ we have $\frac{dy}{dx}=u\,.\,\frac{dv}{dx}+v\,.\,\frac{du}{dx}$.

Example 3.

Find $\frac{d}{dx}\{(x^2+1)(2x+3)\}$.

$$\frac{d}{dx}\{(x^2+1)(2x+3)\}=(x^2+1)\,.\,\frac{d}{dx}(2x+3)+(2x+3)\,.\,\frac{d}{dx}(x^2+1)$$

$$=(x^2+1)\,.\,2+(2x+3)\,.\,2x$$

$$=2x^2+2+4x^2+6x$$

$$=6x^2+6x+2.$$

As a check, $(x^2+1)(2x+3)=2x^3+3x^2+2x+3$,

$$\therefore\ \frac{d}{dx}\{(x^2+1)(2x+3)\}=6x^2+6x+2.$$

This result may be extended to a product consisting of more than two factors.

Thus
$$\frac{d}{dx}(uvw) = \frac{d}{dx}[(uv)(w)]$$
$$= uv \cdot \frac{dw}{dx} + w\frac{d}{dx}(uv)$$
$$= uv \cdot \frac{dw}{dx} + w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]$$
$$= uv \cdot \frac{dw}{dx} + wu\frac{dv}{dx} + vw\frac{du}{dx}.$$

A useful form of this result which shows its extension to any number of factors is obtained by writing it in the form:

$$\frac{\dfrac{d}{dx}(uvw)}{uvw} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx}.$$

IV. *Differentiation of a Quotient.*

If u, v are any functions of x,

$$\mathbf{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}}.$$

With the same notation:

If $y = \frac{u}{v}$, $\qquad y + \delta y = \dfrac{u + \delta u}{v + \delta v}$;

$$\therefore\ \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$
$$= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$$
$$= \frac{vu + v \cdot \delta u - uv - u \cdot \delta v}{v(v + \delta v)};$$
$$\therefore\ \frac{\delta y}{\delta x} = \frac{v \cdot \dfrac{\delta u}{\delta x} - u \cdot \dfrac{\delta v}{\delta x}}{v^2 + v \cdot \dfrac{\delta v}{\delta x} \cdot \delta x}.$$

$\therefore$ when $\delta x \to 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Example 4.

Find $\dfrac{d}{dx}\left(\dfrac{x^2+1}{3x+2}\right)$.

$$\frac{d}{dx}\left(\frac{x^2+1}{3x+2}\right) = \frac{(3x+2)\dfrac{d}{dx}(x^2+1) - (x^2+1)\dfrac{d}{dx}(3x+2)}{(3x+2)^2}$$

$$= \frac{(3x+2)(2x) - (x^2+1)(3)}{(3x+2)^2} = \frac{6x^2+4x-3x^2-3}{(3x+2)^2}$$

$$= \frac{3x^2+4x-3}{(3x+2)^2}.$$

V. $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$.

Now $$\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1.$$

But when δx and $\delta y \to 0$ the limit of $\dfrac{\delta y}{\delta x}$ is $\dfrac{dy}{dx}$ and the limit of $\dfrac{\delta x}{\delta y}$ is $\dfrac{dx}{dy}$.

$$\therefore \frac{dx}{dy} \times \frac{dy}{dx} = 1,$$

i.e. $$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.$$

This result may be deduced as a particular case of VI. Since

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},$$

we have if y is put equal to x

$$\frac{dx}{dx} = \frac{dx}{du} \times \frac{du}{dx},$$

i.e. $$1 = \frac{dx}{du} \times \frac{du}{dx} \text{ or } \frac{dx}{du} = \frac{1}{\dfrac{du}{dx}}.$$

This result is also evident from geometrical considerations.

In each of the figures $\frac{dy}{dx} = \tan\psi$, $\frac{dx}{dy} = \tan\psi'$.

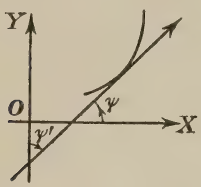

Fig. 126.

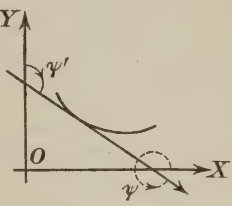

Fig. 127.

In Fig. 126, $\psi + \psi' = \frac{\pi}{2}$ and in Fig. 127, $\psi + \psi' = \frac{5\pi}{2}$,

$$\therefore \text{ in each case } \tan\psi' = \cot\psi \text{ or } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

It should be noted that the sense of the tangent as indicated by the arrow is that in which x is increasing: further, the angle which a line of definite sense makes with $\overrightarrow{OX}$ is measured by the anti-clockwise rotation from OX *towards* OY and the angle it makes with $\overrightarrow{OY}$ by the clockwise rotation from OY *towards* OX.

Example 5.

If $y = x^{\frac{1}{3}}$, find $\frac{dy}{dx}$.

Now $x = y^3$, $\quad \therefore \frac{dx}{dy} = 3y^2 = 3x^{\frac{2}{3}}$,

$$\therefore \frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

VI. *Function of a function.*

We shall illustrate the method by an example. It consists in the application of the formula $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$.

Example 6.

To find $\dfrac{dy}{dx}$ when $y=(ax+b)^{10}$.

Let $u=(ax+b)$, then $y=u^{10}$.

y is here a function of u and u is a function of x; $\therefore$ y is a function of a function of x.

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$, but $\dfrac{dy}{du}=10u^9$ and $\dfrac{du}{dx}=a$;

$$\therefore\ \frac{dy}{dx}=10u^9\,(a)=10a\,(ax+b)^9.$$

Note. $\dfrac{d}{dx}(ax+b)^{10}$ is *not* $10\,(ax+b)^9$ but $10a\,(ax+b)^9$.

This substitution method is of general application to functions of a function.

EXAMPLES XI a

Differentiate with respect to x:

1. $5x^3+2x-4$.
2. $4x^5-5x^4$.
3. $2\frac{1}{2}x^6-1\frac{1}{4}x^4+2\frac{1}{3}$.
4. $(x+2)(x+3)$.
5. $(x^2-x)(2x+1)$.
6. $(2x^2+5x+1)(x^3+x+3)$.
7. $(x+1)(x+2)(x+3)$.
8. $(2x-1)(x^3+1)$.
9. $\dfrac{x+1}{x+2}$.
10. $\dfrac{2x-1}{x^2}$.
11. $\dfrac{3x+4}{5x-3}$.
12. $\dfrac{x^2}{x^2+1}$.
13. $\dfrac{(x+1)^2}{x-1}$.
14. $\dfrac{2x}{(x+1)^2}$.
15. $\dfrac{x^2-2x-5}{3x-2}$.
16. $(2x+3)^5$.
17. $(4-3x)^8$.
18. $(2-x^2)^6$.
19. $(x-3)(x+2)^2$.
20. $\dfrac{x}{(x-1)^2}$.
21. $(ax+b)^2(cx+d)^2$.
22. $\dfrac{(5x-3)^3}{(x+2)^4}$.
23. $\dfrac{(2x-3)^2}{2x-3x^2}$.
24. $(x^3+x-5)^2$.
25. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$.
26. $\dfrac{1}{(x^2+4)^2}$.
27. $\left(\dfrac{1-x}{1+x}\right)^2$.

Derivative of x^n when n is fractional or negative

VII. If n is a positive fraction, $\dfrac{d}{dx}(x^n)=nx^{n-1}$.

Let $n=\dfrac{p}{q}$ where p, q are positive integers.

Let $y = x^n = x^{\frac{p}{q}}$ and put $x^{\frac{1}{q}} = z$ or $x = z^q$,

$$\therefore\ y = x^{\frac{p}{q}} = z^p.$$

Now $\dfrac{dy}{dz} = \dfrac{dy}{dx} \times \dfrac{dx}{dz}$; $\quad \therefore\ \dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dz}}{\dfrac{dx}{dz}} = \dfrac{\dfrac{d}{dz}(z^p)}{\dfrac{d}{dz}(z^q)}$.

$$\therefore\ \frac{dy}{dx} = \frac{pz^{p-1}}{qz^{q-1}} = \frac{p}{q}\, z^{p-q} = \frac{p}{q}\, x^{\frac{p-q}{q}} = \frac{p}{q}\, x^{\frac{p}{q}-1},$$

$$\therefore\ \frac{d}{dx}(x^n) = \frac{dy}{dx} = \frac{p}{q}\, x^{\frac{p}{q}-1} = nx^{n-1}.$$

VIII. If n is any negative number, $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

Let $n = -m$ where m is a positive number.

$$\therefore\ \frac{d}{dx}(x^n) = \frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \dfrac{d}{dx}(1) - \dfrac{d}{dx}(x^m)}{x^{2m}} = \frac{0 - mx^{m-1}}{x^{2m}} \text{ by (VI.)}$$

$$= -mx^{m-1-2m} = -mx^{-m-1}$$

$$= nx^{n-1}.$$

We have now proved that the formula

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

holds for all values of n, positive or negative, fractional or integral.

Example 7.

Find $\dfrac{d}{dx}\left(\dfrac{1}{x^3}\right)$ and $\dfrac{d}{dx}(\sqrt{x})$.

$$\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}.$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

Example 8.

Find $\dfrac{d}{dx}(\sqrt{4-x^2})$.

Put $y=\sqrt{4-x^2}$ and $4-x^2=z$, so that $y=\sqrt{z}$.

$$\begin{aligned}\therefore\ \frac{dy}{dx}&=\frac{dy}{dz}\times\frac{dz}{dx}\\ &=\frac{d}{dz}(z^{\frac{1}{2}})\times\frac{d}{dx}(4-x^2)\\ &=\tfrac{1}{2}z^{-\frac{1}{2}}(-2x)\\ &=-\frac{x}{z^{\frac{1}{2}}}=-\frac{x}{\sqrt{4-x^2}}.\end{aligned}$$

After a little practice it will be found unnecessary to make an actual substitution: the working of this example will then read as follows:

$$\begin{aligned}\frac{d}{dx}(\sqrt{4-x^2})&=\frac{d\sqrt{4-x^2}}{d(4-x^2)}\times\frac{d(4-x^2)}{dx}\\ &=\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\times(-2x)\\ &=-\frac{x}{\sqrt{4-x^2}},\end{aligned}$$

and eventually the first step will be done mentally without actually being written down at all.

EXAMPLES XI b

Differentiate with respect to x the expressions in Examples 1—18:

1. $\dfrac{1}{x^3}$, $x^2-\dfrac{2}{x^2}$, $\dfrac{4}{x}+\dfrac{5}{x^3}$, $\dfrac{10}{x^{10}}$, $\dfrac{1}{x^{2n}}$.

2. $x^{1\cdot5}$, $x^{\frac{2}{3}}$, $\sqrt[3]{x}$, $3x\sqrt{x}$, $\frac{1}{2}\sqrt{x^5}$, $x^{\frac{1}{q}}$, $x^{\frac{2}{n}}$.

3. $\dfrac{1}{\sqrt{x}}$, x^{-3}, $x^{-\frac{1}{3}}$, x^0, $\dfrac{4}{\sqrt[3]{x}}$, $\dfrac{1}{x^{\frac{1}{q}}}$.

4. $x^{-2\cdot3}$, $3x^{-2}$, $\sqrt{\dfrac{5}{x^3}}$, $3x^{\frac{1}{3}}$, $4x^{-\frac{1}{2}}$, $\dfrac{2}{x\sqrt{x}}$, $\sqrt{(5-2x)}$.

5. $\left(x+\dfrac{2}{x}\right)\left(x^2-\dfrac{3}{x^2}\right)$.

6. $(x^{2n}-3)(x^n+1)$.

7. $\dfrac{\sqrt{x}}{1+\sqrt{x}}$.

8. $\dfrac{1}{\sqrt{(3-x)}}$.

9. $\sqrt{(x^2-2x)}$.

10. $\dfrac{1}{\sqrt{(9-x^2)}}$.

11. $\sqrt[3]{(3x^3+8)}$.

12. $\dfrac{x}{\sqrt{(x^2-1)}}$.

13. $x^2\sqrt{(1-x^2)}$.

14. $\sqrt{\left(\frac{1+2x}{1-x}\right)}$. **15.** $x+\sqrt{(1+x^2)}$. **16.** $\{x+\sqrt{(1+x^2)}\}^3$.

17. $\frac{\sqrt{(2x^2-x-5)}}{1+x}$. **18.** $\frac{1}{\sqrt{(x+1)}-\sqrt{x}}$.

19. If $xy^2=4$ express $\frac{dy}{dx}$ in terms of x, and $\frac{dx}{dy}$ in terms of x. What is $\frac{dy}{dx}\times\frac{dx}{dy}$?

20. If $y=\sqrt{a^2-x^2}$ where a is a constant, prove that $y\frac{dy}{dx}+x=0$.

21. If $y=\frac{x}{x+c}$ where c is a constant, prove that $x\frac{dy}{dx}=y(1-y)$.

22. If xy is constant, prove that $\frac{dy}{dx}=-\frac{y}{x}$.

23. Show that $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$.

24. If $v=\frac{ds}{dt}$, show that $\frac{dv}{dt}=\frac{d}{ds}(\frac{1}{2}v^2)$.

25. If $xy=c^2$ where c is constant and if u is any function of x, show that $x\frac{du}{dx}+y\frac{du}{dy}=0$.

26. A pendulum of length l feet makes a complete vibration in t secs. where $t=1{\cdot}11\sqrt{l}$. If the length is measured as 4·5 feet, and if this is correct to 2 significant figures, find approximately the possible error in the time when calculated from this formula.

27. Find the stationary values of $\frac{(1-x)^3}{1-2x}$.

28. A large stone is dropped into a smooth sheet of water at A; the wave-front reaches a point P, y feet from A, after t seconds where $y=\frac{50t}{t+10}$. What is the speed of the wave-front after (i) 1 second, (ii) 10 seconds?

29. For what values of x has the function $(x-2)^{\frac{2}{3}}(x-3)^{\frac{3}{5}}$ a maximum or minimum value?

30. A rectangle is inscribed in a semicircle of radius a so that one side lies on the diameter and the other two corners on the circular arc. Find its maximum area.

31. A, B are two fires 30 feet apart: the fire at A gives out 8 times as much heat as the fire at B. If the intensity of the heat varies inversely as the square of the distance, find what point on AB receives least heat.

32. A light is suspended at a point O, x feet above a table whose top AB is horizontal; A is vertically below O. The intensity of illumination at a point B on the table is proportional to $\frac{OA}{OB^3}$. If $AB=3$ feet, find the height OA in order that the table at B may be illuminated as much as possible.

33. A man is in a boat 2 miles from the nearest point A of a straight sea-coast and wishes to reach a point B on the coast 5 miles from A in the shortest time. He can row 3 miles an hour and walk 4 miles an hour. How far from A should he land?

34. If $y=\frac{x+c}{1+x^2}$ where c is a constant, prove that when y is stationary $2xy=1$.

Implicit Functions

The next example shows how $\frac{dy}{dx}$ may be obtained when x and y are connected by an implicit relation.

Example 9.

To find $\frac{dy}{dx}$, given that $x^2+3xy+y^2=4$.

(i) *From first principles*, if $(x+\delta x, y+\delta y)$ be the coordinates of a point on the curve near (x, y) we have

$$(x+\delta x)^2+3(x+\delta x)(y+\delta y)+(y+\delta y)^2=4,$$

$$\therefore x^2+2x.\delta x+(\delta x)^2+3xy+3x.\delta y+3y.\delta x+3\delta x.\delta y+y^2+2y.\delta y+(\delta y)^2=4.$$

But $x^2+3xy+y^2=4$,

$$\therefore 2x.\delta x+3x.\delta y+3y.\delta x+2y.\delta y+(\delta x)^2+3\delta x.\delta y+(\delta y)^2=0,$$

$$\therefore 2x+3x.\frac{\delta y}{\delta x}+3y+2y.\frac{\delta y}{\delta x}+\delta x+3\frac{\delta y}{\delta x}.\delta x+\left(\frac{\delta y}{\delta x}\right)^2.\delta x=0.$$

$\therefore$ in the limit, when $\delta x \to 0$, we have

$$2x+3x\frac{dy}{dx}+3y+2y\frac{dy}{dx}=0,$$

$$\therefore (3x+2y)\frac{dy}{dx}=-(2x+3y),$$

$$\therefore \frac{dy}{dx}=-\frac{2x+3y}{3x+2y}.$$

(ii) *Differentiating by rule* with respect to x.

$$\frac{d}{dx}(x^2)+3\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=0,$$

$$\therefore\ 2x+3\left(y+x\frac{dy}{dx}\right)+\frac{d}{dy}(y^2)\times\frac{dy}{dx}=0,$$

$$\therefore\ 2x+3y+3x\frac{dy}{dx}+2y\frac{dy}{dx}=0,$$

$$\therefore\ \text{as before } \frac{dy}{dx}=-\frac{2x+3y}{3x+2y}.$$

Parametric Notation

x and y are sometimes expressed each in terms of a third variable, called a *parameter*.

Example 10.

To find $\frac{dy}{dx}$, given that $y=3t^2+t$, $x=5t^3-3$.

Since $\frac{dy}{dx}\times\frac{dx}{dt}=\frac{dy}{dt}$, we have $\frac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Now $\frac{dy}{dt}=\frac{d}{dt}(3t^2+t)=6t+1$ and $\frac{dx}{dt}=\frac{d}{dt}(5t^3-3)=15t^2$,

$$\therefore\ \frac{dy}{dx}=\frac{6t+1}{15t^2}.$$

Example 11.

A bird A is flying horizontally 150 feet above the ground at a speed of 10 ft. per sec. Find the rate at which its distance from C is increasing when $CB=200$ feet where B is the point on the ground vertically below A.

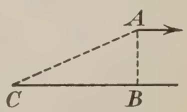

Fig. 128.

AB is constant and $=150$ feet: CB is a variable increasing at the rate of 10 ft. per sec.; let $CB=x$ feet so that $\frac{dx}{dt}=10$ where t is the time measured in seconds.

Let $AC=y$ feet: it is required to find $\frac{dy}{dt}$.

Now $y^2=x^2+150^2$ since $\angle ABC=90°$.

(i) From first principles with the usual notation

$$(y+\delta y)^2=(x+\delta x)^2+150^2.$$

$$\therefore\ y^2+2y\,.\,\delta y+(\delta y)^2=x^2+2x\,.\,\delta x+(\delta x)^2+150^2,$$

but

$$y^2=x^2+150^2,$$

$$\therefore\ 2y\,.\,\delta y+(\delta y)^2=2x\,.\,\delta x+(\delta x)^2.$$

Divide both sides by δt,

$$\therefore\ 2y\frac{\delta y}{\delta t}+\left(\frac{\delta y}{\delta t}\right)^2.\,\delta t=2x\frac{\delta x}{\delta t}+\left(\frac{\delta x}{\delta t}\right)^2.\,\delta t.$$

$\therefore$ in the limit when $\delta t \to 0$ we have

$$2y\frac{dy}{dt}=2x\frac{dx}{dt}.$$

Now when $x=200$, $y^2=200^2+150^2=40{,}000+22{,}500=62{,}500$,

$$\therefore\ y=250,$$

$$\therefore\ \frac{dy}{dt}=\frac{200}{250}\times 10=\underline{8 \text{ ft. per sec.}}$$

(ii) By differentiation, $\dfrac{d}{dt}(y^2)=\dfrac{d}{dt}(x^2)+0.$

But $\dfrac{d}{dt}(y^2)=\dfrac{d}{dy}(y^2)\times\dfrac{dy}{dt}=2y\dfrac{dy}{dt}$ and $\dfrac{d}{dt}(x^2)=\dfrac{d}{dx}(x^2)\times\dfrac{dx}{dt}=2x\dfrac{dx}{dt}.$

$$\therefore\ 2y\frac{dy}{dt}=2x\frac{dx}{dt} \text{ as before.}$$

EXAMPLES XI c

Find $\dfrac{dy}{dx}$ for the relations given in nos. 1—12:

1. $x^2+y^2=4.$ **2.** $2x^2-3y^2=12.$ **3.** $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}.$

4. $x^3+y^3=3xy.$ **5.** $xy(x+y)=c.$ **6.** $x^3y^2=c.$

7. $xy+2x-3y=1.$ **8.** $\sqrt{x}+\sqrt{y}=\sqrt{c}.$

9. $x=at^2$, $y=2at.$ **10.** $x=t^2-1$, $y=(t-1)^2.$

11. $x=\dfrac{3m}{1+m^3}$, $y=\dfrac{3m^2}{1+m^3}.$ **12.** $x=\dfrac{2t}{1+t^2}$, $y=\dfrac{1-t^2}{1+t^2}.$

13. The ends P, Q of a rod PQ of length 10 ins. move on two perpendicular lines OA, OB: if P moves steadily at 3 ins. per sec., find the velocity of Q when Q is 6 ins. from O.

14. A stone is thrown so that after t seconds it has travelled $20t$ feet horizontally and $50t-16t^2$ feet vertically. In what direction is it moving after 1 second?

15. A man standing on a wharf is drawing in the painter of a boat at the rate of 2 ft. per sec. His hands are 6 feet above the level of the bow of the boat. How fast is the boat moving through the water when there are still 10 feet of painter out?

16. If $(x-c)(y-c)=1+c^2$ where c is a constant, prove that

$$\frac{dy}{dx}+\frac{1+y^2}{1+x^2}=0.$$

17. If $s=xy$, $t=x+y$, $x^2+y^2=c^2$ where c is a constant, find $\frac{ds}{dt}$ in terms of x, y.

18. If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, find $\frac{dy}{dx}$.

19. If $xy^2=x^2+4y^2$, find the least positive value of y.

20. A ring is slipping down the rough curve $xy=24$ (unit on each axis 1 foot, y-axis vertical) so that its horizontal velocity is 2 ft. per sec.; what is its vertical velocity when at a height of 6 feet above Ox?

21. One end A of the bounding diameter AB of a semicircular plate moves uniformly at 3 ins. per minute in a slot Oy and the curved rim rolls along a fixed perpendicular line Ox; if $AB=20$ ins., find the rate at which its point of contact P with Ox is moving when $OA=2$ ins.

22. A particle moves s feet in t seconds where $s^3=kt^2$, k being a constant; prove that its acceleration varies as $\frac{1}{s^2}$.

23. A ladder PQ 20 feet long rests with one end P on the ground AP and with the other projecting over a vertical wall AB 12 feet high. The end P is pushed along the ground towards the wall at 2 ft. per sec. What is the *vertical* velocity of Q when P is 5 feet from the wall?

24. A body travels s feet in t secs. where $t=a+bs+cs^2$, a, b, c being constants; prove that its acceleration varies as the cube of its velocity.

25. A rod OP of length l feet is hinged at a point O on the ground and rests against a cylinder of radius a feet which rolls along the ground towards O; if the point of contact Q of the cylinder with the ground is x feet from O when the height of P above the ground is z feet, prove that $z=\frac{2alx}{a^2+x^2}$. If the cylinder is advancing towards O at the rate of a ft. per sec., find the rate at which the height of P is increasing when $OQ=3a$.

26. (i) If $s=at^3$, express, in terms of t, $\frac{ds}{dt}\times\frac{dt}{ds}$; $\frac{d^2s}{dt^2}\times\frac{d^2t}{ds^2}$.

(ii) A body moves s feet in t seconds in such a way that its acceleration varies as the cube of its velocity; prove that $\frac{d^2t}{ds^2}$ is constant.

27. Equal weights of 5 lbs. each are fastened to the ends of a string 30 ins. long which passes over two small smooth pegs A, B in a horizontal line 8 ins. apart. A weight of 6 lbs. is fastened to the middle point of the string. In equilibrium the centre of gravity of the system is at a maximum depth below AB. If the 5 lb. weights are each at a depth of x ins. and the 6 lb. weight at a depth of y ins. below AB, then the depth of the centre of gravity is $\frac{10x+6y}{16}$ ins. Assuming this result, find y in the position of equilibrium.

If the 6 lb. weight is pulled down and released so that it passes through its equilibrium position with a velocity of 8 ins. per sec., what velocity has each 5 lb. weight at this moment?

MISCELLANEOUS EXAMPLES 13—17

M. 13

1. Prove that the sum of the intercepts on the axes made by the tangents to $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ is constant and equal to a.

2. A ship A steams due E. from a port P at 20 knots, setting out at the same moment that another ship B which is steaming at 15 knots towards P is 40 nautical miles due S. of P. Find the distance between the ships at time t hours after A has left P and find for what value of t that distance is least.

3. If the distance s travelled in time t is given by $s=\frac{a}{t}+bt^2$, prove that the acceleration equals $\frac{2s}{t^2}$.

4. ABC is a triangular sheet of thin paper which is being rolled up round AB, a circular rod of radius 1 in., at the rate of 3 turns per sec. Find the rate at which the area ABC is diminishing at the end of the 1st sec., neglecting the increase of radius of the roller due to the rolled up paper.

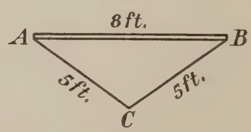

Fig. 129.

5. The sum of the perimeters of two equal squares and a circle is 100 feet. If x is the radius of the circle, draw a graph of A, the total area of the squares and the circle, in terms of x.

When is A least and when greatest? Discuss whether these are maximum and minimum values.

M. 14

1. If O is the origin and P the point (x_1, y_1) on the curve $x^2-y^2=4$, prove that the tangent at P makes the same angle with OX that OP makes with OY.

2. A triangular field ABC has a right angle at A and is divided into two equal parts by a fence PQ of length y cutting AB at P and AC at Q. If $AP=x$, $AB=c$, $AC=b$, prove $y^2=x^2+\dfrac{b^2c^2}{4x^2}$, and find the length of the shortest fence.

3. Sketch the curve $y^2x=(x+1)^2$. Find the abscissa of the point where the tangent is parallel to OX. Show that for values of x greater than this the angle the tangent makes with OX does not exceed the value $\tan^{-1}\dfrac{1}{3\sqrt{3}}$ which occurs when $x=3$.

4. A stone is dropped into a pond and the radius of the outer circular ripple increases steadily at the rate of 5 ft./sec. How rapidly is the area of the disturbed water increasing at the end of 3 secs.?

5. On AB, 6 ins. long, a semicircle is described. C is the mid-point of AB and another semicircle is described on the other side of AB with AC as its diameter. From a point Q on the smaller semicircle QN is drawn perpendicular to AB and produced to meet the other curve in P. Find the maximum value of PQ.

M. 15

1. Find the equations of the tangents to $y=\dfrac{x(x-1)(x+1)}{(x+3)(x-4)}$ where $x=0, +1, -1$. Sketch the curve.

2. A range-finder has a base of d feet which subtends an angle of x seconds at an object whose range is R yards. Show that $x \doteqdot \dfrac{216,000\,d}{\pi R}$. Prove that the error in the range due to a constant small error δx in the angle is proportional to R^2.

3. Two sources of heat at A and B with intensities a and b respectively produce a total intensity at a point P on AB distant x from A given by $I=\dfrac{a}{x^2}+\dfrac{b}{(d-x)^2}$ where $AB=d$. Show that the temperature at P will be least when $\dfrac{d-x}{x}=\sqrt[3]{\dfrac{b}{a}}$.

4. If $pv^\gamma=c$ for a gas, and h the rate at which it absorbs heat is given by $h=\dfrac{1}{\lambda-1}\left\{v\dfrac{dp}{dv}+\lambda p\right\}$, find h in terms of p, λ, γ, and if h is always zero prove $\lambda=\gamma$.

5. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle is proportional to $\dfrac{x}{(a^2+x^2)^{\frac{5}{2}}}$ where x is the distance of the magnet from the plane of the circle. Prove that the force is a maximum when $x=\dfrac{a}{2}$.

M. 16

1. The weight of gas which will flow per second through an orifice from a vessel where it is at pressure p_1 into another vessel where it is at pressure p_2 is proportional to $a^{\frac{1}{\gamma}}\sqrt{1-a^{\frac{\gamma-1}{\gamma}}}$ where $a=\dfrac{p_2}{p_1}$ and $\gamma=1{\cdot}41$. Prove that a maximum quantity will leave a vessel per second when the outside pressure is a little greater than half the inside pressure.

2. If $p=\dfrac{\theta}{v-a}-\dfrac{c}{\theta(v+b)^2}$, where a, b, c, θ are constants, is plotted as a (p, v) curve, p-axis horizontal, prove that for a certain value of θ there is a horizontal inflection at $v=3a+2b$.

3. The distances u and v of an object and its image from a lens whose focal length is f are related by $\dfrac{1}{v}-\dfrac{1}{u}=\dfrac{1}{f}$. If the magnification is $\dfrac{\text{size of image}}{\text{size of object}}$, prove that the magnification along the axis of a small object of width δu is $\left(\dfrac{v}{u}\right)^2$ approximately.

4. If x is the number of gram molecules of a substance A which are transformed by a reaction with another substance B at a time t, then $\frac{x}{a-x}=akt$. Show that the velocity of the reaction varies as the square of the amounts of A and B present at time t where a is the number of gram molecules of A and of B present at the beginning.

5. When compressed air escapes the rate of decrease of pressure P is proportional to the square root of the difference between P and the atmospheric pressure P_0. If $p=P-P_0$, prove $\frac{dt}{dp}=-\frac{1}{k\sqrt{p}}$. If P reduces from 5 atmospheres to 4 in 1 min., find P after 2 mins.

M. 17

1. If a chord of a circle moves across a circle at right angles to a diameter, find the rate of increase of the area of the segment per unit increase of the chord's distance from the end of the diameter.

2. Find the gradient of the tangent to $y=x^2\sqrt{2+x}$ and state its value where the curve meets OX. Find the position of the maximum ordinate and the angle the tangent makes with OX at the point $x=-1$.

3. Water is escaping from the bottom of a cylindrical vessel at a rate $k\sqrt{x}$ where x is the depth of water remaining. If the area of the water surface is A, and if the initial depth is C, find the time it takes to empty.

4. Prove that $f(x+h) \simeq f(x)+hf'(x)$. One root of $3x^2-x-1=0$ is approximately 0·77. Suppose a more correct value to be $(0{\cdot}77+h)$ and, by using the above relation, find h and hence a better approximation to the root.

5. A body moves so that its velocity after travelling s feet is v ft.-sec. where $v^2=a^2-s^2$; find its acceleration in terms of s.

CHAPTER XII

DERIVATIVES OF THE TRIGONOMETRICAL FUNCTIONS

It is essential that the student should be familiar with the Radian method of measuring angles and with the relation between radians and degrees, before attempting to differentiate the Trigonometrical functions.

Derivative of $\sin x$

Draw a rough graph of $y = \sin x$ from $x = 0$ to $x = 2\pi$ radians. Let (x, y) be the coordinates of P, then $y\,(NP)$ represents the

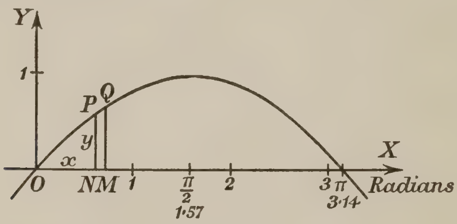

Fig. 130.

sine of the angle whose radian measure is $x\,(ON)$, i.e. $y = \sin x$, and since MQ is the sine of the angle whose radian measure is OM we have

$$y + \delta y = \sin(x + \delta x),$$

$$\therefore \ \delta y = \sin(x + \delta x) - \sin x$$

$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right).$$

$$\therefore \ \frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\dfrac{\delta x}{2}\right)}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\dfrac{\delta x}{2}\right)}{\dfrac{\delta x}{2}}.$$

Now as $\delta x \to 0$, the limit of

$$\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

equals the limit to which $\dfrac{\sin\theta}{\theta}$ tends *where θ is measured in radians* if $\theta \to 0$.

This limit was shown to be 1 in Part I, p. 20, hence

$$\frac{d}{dx}(\sin x) = \frac{dy}{dx} = \mathop{\mathrm{Lt}}_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \mathop{\mathrm{Lt}}_{\delta x \to 0} \left\{ \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}} \right\}$$

$$= \cos x.$$

In particular we notice that if Fig. 130 represents the graph of $y = \sin x$, with the same units for OX, OY, then for $x = 0$, the gradient $= \cos 0 = 1$, and so the slope is $\dfrac{\pi}{4}$; this means that $\sin x$ increases at the same rate as x (in radians) at the origin: also at $x = \dfrac{\pi}{2}$, the gradient $= \cos\dfrac{\pi}{2} = 0$ and we have a maximum.

Note. It is most important to realise that the relation

$$\frac{d}{dx}(\sin x) = \cos x$$

is true, *only if* x is measured in radians.

Consider $\dfrac{d}{dz}(\sin z^\circ)$ where z° means z degrees.

Now $z^\circ = \dfrac{\pi z}{180}$ radians $= t$ radians, say.

$$\therefore\ \frac{d}{dz}(\sin z^\circ) = \frac{d}{dz}(\sin t) = \frac{d(\sin t)}{dt} \times \frac{dt}{dz}.$$

But $\dfrac{d}{dt}(\sin t) = \cos t$ and $\dfrac{dt}{dz} = \dfrac{\pi}{180}$.

$$\therefore \quad \frac{d}{dz}(\sin z^\circ) = \cos t \times \frac{\pi}{180} = \frac{\pi}{180}\cos z^\circ.$$

By employing radians instead of degrees, when differentiating, we avoid this awkward numerical factor.

Derivative of $\cos x$

Fig. 131 represents the graph of $y = \cos x$ between 0 and $\dfrac{\pi}{2}$, x being measured in radians. Since $\cos x$ decreases as x increases it is clear that $\dfrac{d}{dx}(\cos x)$ is negative, if x is acute.

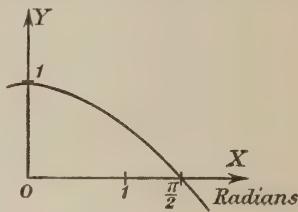

Fig. 131.

As before,

$$\delta y = \cos(x + \delta x) - \cos x$$

$$= -2\sin\left(x + \frac{\delta x}{2}\right) . \sin\frac{\delta x}{2}.$$

$$\therefore \quad \frac{\delta y}{\delta x} = -2\sin\left(x + \frac{\delta x}{2}\right) . \frac{\sin\frac{\delta x}{2}}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right) . \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

$$\therefore \quad \frac{d}{dx}(\cos x) = \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}}\left(\frac{\delta y}{\delta x}\right) = -\sin x.$$

Note. (i) We can also obtain this result as follows:

$$\frac{d}{dx}[\cos x] = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \frac{d\sin\left(\frac{\pi}{2} - x\right)}{d\left(\frac{\pi}{2} - x\right)} \times \frac{d\left(\frac{\pi}{2} - x\right)}{dx}$$

$$= \cos\left(\frac{\pi}{2} - x\right) \times (-1) = -\sin x.$$

(ii) It is important to observe as before that the relation

$$\frac{d}{dx}(\cos x) = -\sin x$$

is true *only if* x is measured in radians.

By the same method as above we could show that

$$\frac{d}{dz}(\cos z^\circ) = -\frac{\pi}{180}\sin z^\circ.$$

In future we shall assume, unless otherwise stated, that all angles are measured in radians.

Derivatives of $\sin(ax+b)$ and $\cos(ax+b)$

(i) Let $y = \sin(ax+b)$ and $ax+b=u$,

$$\therefore\ y = \sin u,$$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{d}{du}(\sin u) \times \frac{d}{dx}(ax+b)$$

$$= \cos u \times a = a\cos(ax+b).$$

(ii) Similarly

$$\frac{d}{dx}[\cos(ax+b)] = \frac{d\cos(ax+b)}{d(ax+b)} \times \frac{d(ax+b)}{dx}$$

$$= -\sin(ax+b) \times a = -a\sin(ax+b).$$

Alternative Geometrical Method

The derivatives of $\sin\theta$ and $\cos\theta$ may also be obtained as follows:

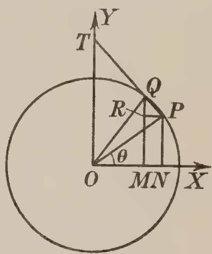

Fig. 132.

Draw a circle, centre the origin O, radius unity.

Let $\angle NOP = \theta$ radians, $\angle POQ = \delta\theta$ radians. R is the foot of the perpendicular from P to QM.

Then $\qquad y = NP = \sin\theta,$

$$y + \delta y = MQ = \sin(\theta + \delta\theta).$$

$$\therefore\ \frac{d}{d\theta}(\sin\theta) = \frac{dy}{d\theta} = \underset{\delta\theta \to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta\theta}$$

$$= \mathrm{Lt}\,\frac{MQ - NP}{\text{arc } PQ} = \mathrm{Lt}\,\frac{RQ}{\text{arc } PQ} = \mathrm{Lt}\,\frac{RQ}{\text{chord } \overline{PQ}} \times \frac{\text{chord } \overline{PQ}}{\text{arc } PQ}.$$

Now, if PQ cuts OY at T,

$$\frac{RQ}{PQ} = \cos RQP = \cos OTP.$$

But when $\delta\theta \rightarrow 0$, TP becomes the tangent at P and

$$\angle OTP = \frac{\pi}{2} - \angle TOP = \theta,$$

$$\therefore \ \text{Lt} \frac{RQ}{PQ} = \cos\theta.$$

Also $\text{Lt} \dfrac{\text{chord } PQ}{\text{arc } PQ} = 1,$

$$\therefore \ \frac{d}{d\theta}(\sin\theta) = \cos\theta,$$

and similarly

$$\frac{d}{d\theta}(\cos\theta) = \frac{dx}{d\theta} = \underset{\delta\theta \rightarrow 0}{\text{Lt}} \frac{\delta x}{\delta\theta} = \text{Lt} \frac{OM - ON}{\text{arc } PQ} = -\text{Lt} \frac{MN}{\text{arc } PQ}$$

$$= -\text{Lt} \frac{RP}{\text{chord } PQ} \times \frac{\text{chord } PQ}{\text{arc } PQ} \qquad \text{since } MN = RP,$$

but $\dfrac{RP}{PQ} = \sin RQP = \sin OTP \rightarrow \sin\theta$ when $\delta\theta \rightarrow 0$,

$$\therefore \ \frac{d}{d\theta}(\cos\theta) = -\sin\theta.$$

Derivative of $\tan x$

If $y = \tan x$, $\qquad y + \delta y = \tan(x + \delta x)$,

$$\therefore \ \delta y = \tan(x + \delta x) - \tan x$$

$$= \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\,.\,\cos x}$$

$$= \frac{\sin\{(x + \delta x) - x\}}{\cos(x + \delta x)\,.\,\cos x} \qquad \text{cf. } \sin(A - B)$$

$$= \frac{\sin\delta x}{\cos(x + \delta x)\,.\,\cos x}.$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{1}{\cos(x + \delta x)\,.\,\cos x} \cdot \frac{\sin\delta x}{\delta x},$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x.$$

This result shows that the gradient of $\tan x$ is always positive, as is clear from a graph of the function.

Alternative Method.

$$y = \tan x = \frac{\sin x}{\cos x}$$

may be differentiated by using the rule for a quotient,

$$\therefore \frac{dy}{dx} = \frac{\cos x (\cos x) - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivative of $\operatorname{cosec} x$

Proceeding as before we have

$$\begin{aligned} \delta y &= \operatorname{cosec}(x + \delta x) - \operatorname{cosec} x \\ &= \frac{1}{\sin(x + \delta x)} - \frac{1}{\sin x} \\ &= \frac{\sin x - \sin(x + \delta x)}{\sin(x + \delta x) \,.\, \sin x} \\ &= -\frac{2\cos\left(x + \frac{\delta x}{2}\right) . \sin\frac{\delta x}{2}}{\sin(x + \delta x) \,.\, \sin x}. \end{aligned}$$

$$\therefore \frac{\delta y}{\delta x} = -\frac{\cos\left(x + \frac{\delta x}{2}\right)}{\sin(x + \delta x) \,.\, \sin x} \,.\, \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}},$$

$$\therefore \frac{dy}{dx} = -\frac{\cos x}{\sin x \,.\, \sin x} = \underline{-\cot x \,.\, \operatorname{cosec} x.}$$

Here again the derivative is negative since $\operatorname{cosec} x$ decreases when x increases from 0 to $\frac{\pi}{2}$.

Alternative Method.

$$y = \operatorname{cosec} x = (\sin x)^{-1}.$$

Let $u = \sin x$, then $y = u^{-1}$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times \cos x = -\frac{\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x.$$

Derivatives of $\sec x$ **and** $\cot x$

The method is similar to that used for the other functions and the details are left to the student: we find

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Summary.

y	$\frac{dy}{dx}$ (x in radians)	$\frac{dy}{dx}$ (x in degrees)
$\sin x$	$\cos x$	$\frac{\pi}{180}\cos x$
$\cos x$	$-\sin x$	$-\frac{\pi}{180}\sin x$
$\tan x$	$\sec^2 x$	$-\frac{\pi}{180}\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$-\frac{\pi}{180}\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$	$\frac{\pi}{180}\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$	$-\frac{\pi}{180}\operatorname{cosec}^2 x$

The results given in this table include the cases where the angle is given in degrees in order to emphasise the differences that occur when degrees are used instead of radians. But the student should always work in radian measure (and so use the simpler formulae) whenever possible.

Example 1.

Differentiate $\sin(2x+4)$.

$$\begin{aligned}\frac{d}{dx}\sin(2x+4) &= \frac{d\sin(2x+4)}{d(2x+4)} \times \frac{d(2x+4)}{dx} \\ &= \cos(2x+4) \times 2 \\ &= 2\cos(2x+4).\end{aligned}$$

Example 2.

Differentiate $\tan^2(3x+2)$.

Let $y=\tan^2(3x+2)$ and put $u=\tan(3x+2)$, then $y=u^2$.

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$=2u\times\sec^2(3x+2)\times 3$$
$$=6\sec^2(3x+2)\tan(3x+2).$$

Or thus:

$$\frac{d}{dx}[\tan^2(3x+2)]=\frac{d[\tan^2(3x+2)]}{d[\tan(3x+2)]}\times\frac{d[\tan(3x+2)]}{d(3x+2)}\times\frac{d(3x+2)}{dx}\quad\text{(done mentally)}$$
$$=2\tan(3x+2)\,.\,\sec^2(3x+2)\,.\,3.$$

EXAMPLES XII a

Differentiate with respect to x the expressions in Ex. 1—16:

1. $\sin(3x+4)$, $\cos 3x$, $\sin(2-x)$.
2. $\tan 4(x-2)$, $\cos(a-x)$, $\sin\frac{2\pi}{3}(x+4)$.
3. $\sec 2x$, $\sec(3-x)$, $\tan(4x-3)$.
4. $\operatorname{cosec} 4x$, $\sin\frac{2\pi}{a}(x-b)$, $\operatorname{cosec}(3x-1)$.
5. $\cot ax$, $\cot(2-x)$, $\tan 3(2-x)$.
6. $\sin^2 x$, $\cos^2 2x$, $\tan^2(x-2)$.
7. $\sin^2(3x-4)$, $\cos^3 3x$, $\operatorname{cosec}^2(x-2)$.
8. $\sec^2(ax-b)$, $\operatorname{cosec}^3 2x$, $\cot(x^{\frac{1}{2}})$.
9. $x\sin x$, $x^2\cos x$, $x\sin x+\cos x$.
10. $\tan x-x$, $x\tan x$, $\cos 2x\sin x$.
11. $\sin x\sin 2x$, $\sin ax\cos bx$, $\cos px\cos qx$.
12. $\frac{1}{2}x+\frac{\sin 2x}{4}$, $\tan 2x\cos x$, $x^2\operatorname{cosec} x$.
13. $\frac{\sin x}{x}$, $\frac{1-\sin x}{1+\sin x}$, $\frac{1}{1+\sin x}$.
14. $\frac{\tan x-1}{\sec x}$, $\cos\frac{a}{x}$, $\sqrt{\sin 2x}$.
15. $\sec^2\frac{x}{2}$, $\tan^2\frac{x}{2}$, $\frac{\cos x}{\sin x+\cos x}$.
16. $\sin(2x^\circ)$, $\cos^2(x^\circ)$, $\tan(\frac{1}{2}x^\circ)$.

Further illustrations

In effecting the inverse process of integration we notice that since

$$\frac{d}{dx}(\sin nx) = n \cos nx,$$

the solution to the equation

$$\frac{dy}{dx} = \cos nx$$

is given by $$y = \frac{1}{n}\sin nx + c,$$

where c is a constant.

In order to integrate powers of $\sin x$ or $\cos x$ we first express them as functions of multiple angles of the first degree.

Example 3.

If $\dfrac{dy}{dx} = \sin^2 x$, find y.

We have $$\frac{dy}{dx} = \tfrac{1}{2}(1 - \cos 2x) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

But $$\frac{d}{dx}(\sin 2x) = 2\cos 2x,$$

$$\therefore\ y = \frac{x}{2} - \tfrac{1}{4}\sin 2x + c,$$

where c is a constant.

Example 4.

A bird A is flying horizontally 150 ft. above the ground at a speed of 10 ft./sec. Find the rate at which its distance from C is increasing when $CB = 200$ ft. (cf. p. 155).

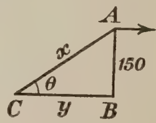

Fig. 133.

Let $A\hat{C}B$ be θ radians, and CA be x.

Then $x = 150 \operatorname{cosec} \theta$,

$$\therefore\ \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$= -150 \operatorname{cosec} \theta \cot \theta \times \frac{d\theta}{dt} \quad \text{.................. (i).}$$

If $CB = y$, then $y = 150 \cot \theta$,

$$\therefore\ \frac{dy}{dt} = -150 \operatorname{cosec}^2 \theta \,.\, \frac{d\theta}{dt} \quad \text{...................... (ii).}$$

But when $y=200$, $\dfrac{dy}{dt}=10$ and $CA=250$,

$$\therefore \ \operatorname{cosec}\theta=\tfrac{250}{150}=\tfrac{5}{3},$$

$$\therefore \ 10=-150\times\tfrac{25}{9}\times\frac{d\theta}{dt} \qquad \text{from (ii)},$$

$$\therefore \ \frac{d\theta}{dt}=-\tfrac{3}{125} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

$$\therefore \ \frac{dx}{dt}=-150\times\tfrac{5}{3}\times\tfrac{4}{3}\times(-\tfrac{3}{125}) \qquad \text{from (i)}$$

$$=\underline{8 \text{ ft./sec.}}$$

From (iii) we see that at this moment the rate of diminution of θ is

$\tfrac{3}{125}$ rad./sec.

Example 5. *Motion of a crank.*

If OP is a crank rotating about O and PQ is the connecting-rod, then the velocity of Q at any time t is $\dfrac{dx}{dt}$.

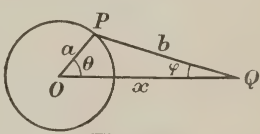

Fig. 134.

But $\quad x=a\cos\theta+b\cos\phi$,

$$\therefore \ \frac{dx}{dt}=a\frac{d}{dt}(\cos\theta)+b\frac{d}{dt}(\cos\phi)$$

$$=a\frac{d}{d\theta}(\cos\theta)\times\frac{d\theta}{dt}+b\frac{d}{d\phi}(\cos\phi)\frac{d\phi}{dt},$$

or

$$\dot{x}=-a\sin\theta\,\dot{\theta}-b\sin\phi\,\dot{\phi}.$$

We can thus obtain the velocity of Q in terms of the angular velocities of OP and PQ.

EXAMPLES XII b

1. Find y if (i) $\dfrac{dy}{dx}=\cos 2x$, (ii) $\dfrac{dy}{dx}=\sin 3x$.

2. Find y if (i) $\dfrac{dy}{dx}=\sec^2 2x$, (ii) $\dfrac{dy}{dx}=\tan^2 2x$.

3. Find y if (i) $\dfrac{dy}{dx}=\sin ax\cos bx$, (ii) $\dfrac{dy}{dx}=\cos^2 x$.

4. Prove that $\quad y=a\cos nx+b\sin nx$

is a solution of the equation $\dfrac{d^2y}{dx^2}+n^2y=0.$

5. Prove that $\quad y=a\sin(nx+a)$

is a solution of $\quad \dfrac{d^2y}{dx^2}+n^2y=0.$

6. If $y=\tan x$, prove that

$$\frac{d^2y}{dx^2}=2y\frac{dy}{dx}.$$

7. Find the maximum and minimum values of $4\cos x+3\sin x$.

8. OX and OY are two rods at right angles. A rod AB 2 ft. long moves with one end A on OY and the other end B on OX. If A is descending towards O at the rate of 2 ft./sec. when AB makes an angle of 30° with OX, find the rate at which B is moving.

9. Calculate $\sin(x+30')-\sin(x-30')$ for $x=0, 20°, 40°, 60°$ and show that the values are approximately proportional to the corresponding values of $\cos x$. If the expression equals $k\cos x$, explain why k is approximately the same as $\frac{\pi}{180}$.

10. A point P describes a circle, centre O, radius a, with uniform angular velocity ω, starting from A, the end of the diameter $A'OA$. PN is the perpendicular from P to $A'A$ after a time t. If $ON=x$, find the velocity of N and also its acceleration in terms of ω, a, t.

11. If x is the deflection in a tangent galvanometer and a given small error is made in reading the deflection, show that the *percentage* error in the current C is proportional to $(\tan x+\cot x)$ where $C=k\tan x$.

12. A particle moves in a straight line so that its distance x from a fixed point in the line at time t is given by $x=a\cos(nt+\epsilon)$; prove that its retardation is proportional to x.

13. Show that $\frac{1+x\tan x}{x}$ is a minimum when $x=\cos x$. From the tables show that $x \doteqdot 0{\cdot}739$.

14. Show that the maximum value of $a\sin x+b\cos x$ is $+\sqrt{a^2+b^2}$ and the minimum value is $-\sqrt{a^2+b^2}$. Also solve this question by writing $a\sin x+b\cos x$ in the form $k\sin(x+\theta)$ where $\theta=\tan^{-1}\frac{b}{a}$.

15. A string, one end of which is fastened at A, passes over a smooth pulley B and carries a weight W at its other end: a smooth ring C (see Fig. 135) slides on the string: AB is horizontal and $=2a$: $AC=CB=x$, $\angle CBA=\theta$. If W descends with velocity v, find the rate of change of θ and the rate of ascent of C, each in terms of a, v, θ, and evaluate each when $\theta=\frac{\pi}{4}$.

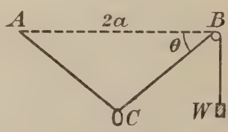

Fig. 135.

16. Fig. 136 represents a rod AB hinged at B and a rod CD hinged to AB at C: the end D slides in the groove BE. If

$$AC = CB = CD = 2 \text{ ft.},$$
$$\angle BCD = \theta°,$$

find the vertical rate of descent of A and the speed of D when $\theta = 90°$, given that θ is increasing at the uniform rate of 1° per sec.

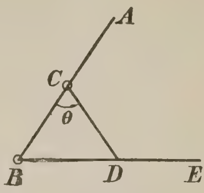

Fig. 136.

Derivatives of Inverse Trigonometrical Functions

If $x = \sin y$, then $y = \sin^{-1} x$ where $\sin^{-1} x$ is written for "the angle whose sine is x."

If we are given $x = \cdot 5$ (say), then y may be $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$ ...; it is therefore a multiple-valued function of x from $x = -1$ to $+1$ both inclusive.

The graph of $y = \sin^{-1} x$ shown in Fig. 137 is obviously the same curve as $x = \sin y$.

If we have the graph of $y = \sin x$ already drawn we can obtain the graph of $y = \sin^{-1} x$ by taking the image of the graph of $y = \sin x$ in the line $y = x$.

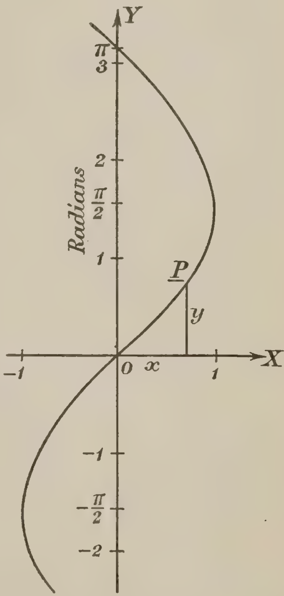

Fig. 137.

Derivative of $\sin^{-1} x$

Let $\qquad y = \sin^{-1} x,$

then $\qquad x = \sin y,$

$$\therefore \frac{dx}{dy} = \cos y,$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\pm\sqrt{1 - \sin^2 y}} = \frac{1}{\pm\sqrt{1 - x^2}}.$$

We see from the graph that the gradient is positive between $y = -\frac{\pi}{2}$ and $+\frac{\pi}{2}$. These values of y are called its *Principal Values*

and the derivatives for these values of x will be given by taking the positive sign of the square root.

$$\therefore \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Derivative of $\cos^{-1} x$

Fig. 138 represents the graph of $y=\cos^{-1} x$; then, proceeding as before, we obtain

$$\frac{dy}{dx} = -\frac{1}{\pm\sqrt{1-x^2}}.$$

The principal values of $\cos^{-1} x$ are those between $y=0$ and π: for these values the gradient of the function is negative and we obtain them by again taking the positive value of the square root.

$$\therefore \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

Note. Since

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},$$

$$\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = 0.$$

$$\therefore \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{d}{dx}(\sin^{-1} x).$$

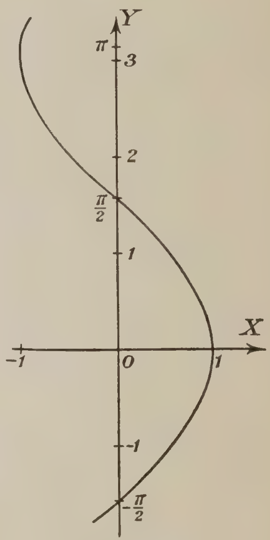

Fig. 138.

Derivative of $\tan^{-1} x$

A similar procedure gives us

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Here there is no ambiguity of sign, since the gradient is always positive. The principal values of $\tan^{-1} x$ are those from

$$y = -\frac{\pi}{2} \text{ to } +\frac{\pi}{2}.$$

If $y = \sin^{-1}\frac{x}{a}$ we have, by putting $z = \frac{x}{a}$,

$$y = \sin^{-1} z$$

and
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{1}{\sqrt{1-z^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2 - x^2}}.$$

Similarly
$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}.$$

But
$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}.$$

These results are chiefly of importance as a clue to methods of integration, and further reference to them will be found in Chapter XIV.

EXAMPLES XII c

Differentiate with respect to x the expressions in Ex. 1—12:

1. $\sin^{-1}(2x)$. **2.** $\cos^{-1}(3x)$. **3.** $\tan^{-1}(4x)$. **4.** $\sin^{-1}\left(\frac{x}{3}\right)$

5. $\frac{1}{x}\sin^{-1}(x)$. **6.** $x^2\cos^{-1}(x)$. **7.** $\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)$. **8.** $\sec^{-1}\left(\frac{x}{a}\right)$.

9. $\cot^{-1}\left(\frac{x}{a}\right)$. **10.** $\tan^{-1}(ax) + \cot^{-1}(ax)$.

11. $(1+x^2)\tan^{-1}x - x$. **12.** $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

13. If $\frac{dy}{dx} = \frac{1}{a^2+x^2}$ and if $x = a\tan\theta$, find $\frac{dy}{d\theta}$.

14. If $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$ and if $x = a\sin\theta$, find $\frac{dy}{d\theta}$.

15. If $\frac{dy}{dx} = \frac{1}{4+x^2}$, find y.

16. If $\frac{dy}{dx} = \frac{1}{\sqrt{(9-x^2)}}$, find y.

17. If $\frac{dy}{dx} = \frac{1}{x^2+2x+10}$, find y.

18. Sketch the graph of $y = \cos x$ from $x = 0$ to $x = 2\pi$; sketch the reflection of this in the line $y = x$. If the equation of this is $y = f(x)$, what is $f(x)$?

REVISION PAPERS 12—17

R. 12

1. If $y=\dfrac{\cos ax}{1+x}$, prove that

$$\frac{d^2y}{dx^2}+\frac{2}{1+x}\frac{dy}{dx}+a^2y=0.$$

2. Find the area of the loop of $y^2=x(x-1)^2$.

3. Differentiate:

$$\text{(i) } \frac{x+1}{\sqrt{x}};\quad \text{(ii) } \sqrt{(x^n+2)};\quad \text{(iii) } \frac{\sin x}{1-\cos x}.$$

4. A, B are two points on the same side of the line CD; AH, BK are the perpendiculars from A, B to CD; P is a point on HK; if $AH=1$, $HK=9$, $BK=2$, $HP=x$, find the value of x for which $AP+PB$ is least and prove that in this case $\angle APH=\angle BPK$.

5. (i) Find y if $\dfrac{dy}{dx}=\sin 2x \sin 3x$;

(ii) If $y=\tan^{-1}x$ and $z=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, find $\dfrac{dy}{dz}$.

R. 13

1. If $y=\dfrac{a\cos kx+b\sin kx}{x}$, prove that

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+k^2y=0.$$

2. A cistern without a lid has a square base: the area of the metal used for its base and sides is 300 sq. ft. What is its maximum volume?

3. Find the position of the centre of gravity of the area bounded by the curve $y^2=x(x-1)^2$ between $x=0$ and $x=1$.

4. Differentiate:

$$\text{(i) } \frac{1}{4-x^3};\quad \text{(ii) } x-\sqrt{(1+x^2)};\quad \text{(iii) } \sec^3 x.$$

5. If $xy^3=8$, express $\dfrac{dy}{dx}$ in terms of x and $\dfrac{dx}{dy}$ in terms of x. What is $\dfrac{dx}{dy}\times\dfrac{dy}{dx}$?

R. 14

1. A circular cylinder is closed at one end and open at the other: its total surface is 10 sq. ins. What is its maximum volume?

2. What is the shape of the curve

$$y^2 = x^2(16 - x^2)?$$

It is rotated about the x-axis: what is the volume of the solid so generated?

3. Differentiate:

$$\text{(i) } \frac{2-x}{1+x^2}; \quad \text{(ii) } \sec^{-1}(2x); \quad \text{(iii) } \cos 3x^\circ.$$

4. A point is moving round a vertical circle at the uniform rate of 10 revolutions per minute; its shadow is projected vertically downwards on to the ground; find the speed of its shadow when it is vertically below the mid-point of a horizontal radius, if the radius is 6″.

5. The mixture inside the cylinder of a petrol engine is compressed according to the law $pv^{1\cdot4}$ = constant: at the beginning of the stroke $p=20$, $v=60$; at the end of the stroke $v=15$. Find the work done in compressing the mixture.

R. 15

1. If $x = \dfrac{2t}{1+t^2}$, $y = \dfrac{1-t^2}{1+t^2}$, express $\dfrac{dy}{dx}$ in terms of t.

2. A particle can slide along the curve $x^2 + 2y^2 = 3$: it is placed at the point (1, 1): if in a small displacement its x-coordinate is increased by 0·1, what is the approximate change in its y-coordinate?

3. Differentiate: (i) $\sin(\sqrt{x})$; (ii) $\sqrt{(\sin x)}$.

4. Find the area of the segment cut off from the parabola

$$y = x(1-x)$$

by the line $x = 4y$.

5. Draw a square $ABCD$ and a line parallel to DC cutting AD, BC at P, Q; $ABCD$ represents a uniform sheet of metal of side 10″ and $PDCQ$ is a strip of similar metal attached to it. Find the length of PD if the centre of gravity of the composite sheet is at a maximum distance from AB.

R. 16

1. By putting $y=tx$, express the x, y coordinates of any point on the curve $x^3+y^3=3xy$ in terms of t, and find the value of t for which the tangent is parallel (i) to the x-axis, (ii) to the y-axis.

2. Differentiate: (i) $\dfrac{x^n}{\underline{|n}}$; (ii) $\sin^2 x+\cos^2 x$.

If $y=\sin(ax)$, find $\dfrac{d^4y}{dx^4}$.

3. If a wheel of radius a is rolling along the ground and if ω is its angular velocity, the position of a point on the rim after t secs. is given by $x=a[\omega t-\sin(\omega t)]$, $y=a[1-\cos(\omega t)]$, find the velocity of the point at that moment.

4. Fig. 139 represents a T-shaped strip of metal: if $AD=6''$, $AB=2''$, $EF=4''$, $FG=1''$, find the radius of gyration (i) about AB, (ii) about AD.

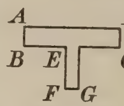

Fig. 139.

5. The power required to drive an aeroplane is measured by $a\sqrt{\theta}+\dfrac{b}{\sqrt{\theta^3}}$, where θ is the angle of attack and a, b are constants; for what value of θ is the power least?

R. 17

1. (i) If $\sin y=\tan x$, express $\dfrac{dy}{dx}$ in terms of x.

(ii) If $y=\cos x$, prove that $\dfrac{d^ny}{dx^n}=\cos\left(x+\dfrac{n\pi}{2}\right)$.

2. If the area of a triangle is calculated from the formula $\Delta=\frac{1}{2}bc\sin A$ and if b, c are measured correctly but A is taken as $60°$ with a possible error of $5'$, calculate the possible percentage error in Δ.

3. Evaluate:

$$\text{(i)}\ \int_0^1 x(1-x)^3\,dx;\quad \text{(ii)}\ \int_1^2 \frac{dx}{x^{0\cdot 3}};\quad \text{(iii)}\ \frac{d}{dx}\sin\left(\frac{1}{x}\right).$$

4. A pressure of 25 lbs. will keep a certain spiral spring compressed through 3 ins.; how much work is needed to compress it an extra inch?

5. A closed vessel is such that when the depth of water in it is x feet, the volume of water is $3\sin^2 x+2\sin 2x$ cu. ft., the angles being in radians. Show that the greatest height of the vessel is about 13·3 ins. and find the area of the cross-section at a height of 3 ins.

MISCELLANEOUS EXAMPLES 18—23

M. 18

1. A small revolving mirror at A reflects a spot of light on to a screen 50 cms. away. If the mirror is rotating at the rate of 10° per second, find the speed of the spot of light when it is 10 cms. from its nearest position to A.

2. P is any point on the arc of a semicircle APB (see Fig. 140); BM is perpendicular to the tangent at P and MR is perpendicular to AB. If $\angle BAP=\theta$, $AB=2a$, find MR and determine for what value of θ it is a maximum.

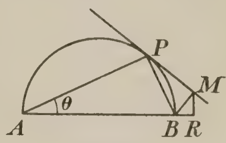

Fig. 140.

3. If the sides a and b of a triangle are measured correctly, but there is an error of δC in the angle C, prove that the error in the area is $\frac{1}{2}ab\cos C \,.\, \delta C$. Find the error in the area if $a=325{\cdot}0$, $b=245{\cdot}0$, $C=60°$, $\delta C=6'$.

4. If $V=a\sin(\omega t-nx)$, calculate $\dfrac{d^2V}{dt^2}$ if a, ω, n, x are constants and $\dfrac{d^2V}{dx^2}$ if a, ω, n, t are constants, and find their ratio.

5. In Rapson's steering gear the tiller AB is worked by a carriage CD which runs on rails PQ, $P'Q'$, backwards and forwards athwart the ship, the carriage being pulled by a chain MN which is kept parallel to the rails. The carriage carries a swivelling block EF having a hole in it, into which slides the end of the tiller. B is the axle of the rudder and θ is the angle GBH which the tiller makes with the fore and aft line. If the carriage moves a small distance δx, prove that the angle turned through by the tiller $=\dfrac{\delta x}{a}\cos^2\theta$ radians approximately.

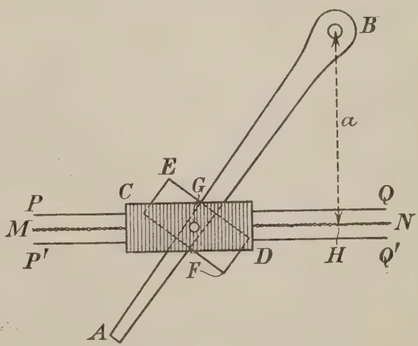

Fig. 141.

(Army.)

M. 19

1. If the distance x ft. a body has travelled in time t secs. is given by $x=1+3\sin 2t-\cos 2t$, find the value of t at which the body first comes to rest and find its acceleration at that time.

2. For a tangent galvanometer $C=\kappa\tan\theta$ where C is the current and θ the deflection. Prove that the proportional error in C due to a given error of reading is least when $\theta=45°$.

3. Find by the Calculus the value of sec 60° 1′, given sec 60° = 2.

4. For an electric circuit

$$V=RC+L\frac{dC}{dt} \text{ and } C=a\sin\kappa t;$$

prove
$$V=\sqrt{(Ra)^2+(La\kappa)^2}\sin(\kappa t+\theta),$$

where
$$\tan\theta=\frac{L\kappa}{R}.$$

5. A rectangular box whose base is a square of side a ft. and whose height is h ft. contains water. It is tilted about BC at a constant angular velocity ω rads./sec. Find the rate at which the wetted surface of AF diminishes up to $\theta=\tan^{-1}\dfrac{h}{a}$ and the surface of AC diminishes after that value.

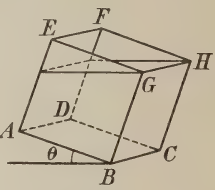

Fig. 142.

M. 20

1. Show that the point $x=a\cos\theta$, $y=b\sin\theta$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Find the gradient of the tangent at this point in terms of θ.

2. A rod AB of length 16 cms. rests between a wall AD and a smooth peg C, 1 cm. from the wall, and makes an angle θ with the horizontal. Prove that the height of G, the mid-point of the rod above the peg, is

$$8\sin\theta-\tan\theta$$

and find for what value of θ this height is a maximum.

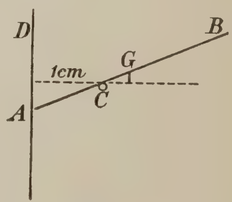

Fig. 143.

3. If $y=uv$, prove that the percentage increase in y due to small increments δu and δv is approximately

$$\left(\frac{\delta u}{u}+\frac{\delta v}{v}\right)100.$$

The intensity of illumination at a point A of a surface due to a source of light at $P=\kappa\,\frac{\cos\theta}{r^2}$, where $r=AP$ and θ is the angle the normal to the surface at A makes with AP. Prove that the percentage increase of illumination at a point near A where r is increased by δr and θ by $\delta\theta$ radians is

$$-100\left(\tan\theta\,\delta\theta+\frac{2\delta r}{r}\right).$$

Evaluate when $\theta=45^\circ$, $r=30$ cms., $\delta\theta=1^\circ$, $\delta r=1$ cm. (Army.)

4. P the end of a crank moves uniformly round a circle of radius a, centre O. Q, the other end of the connecting rod PQ, of length b, moves along a straight line through O. If OP makes an angle θ with OQ, when QP makes an angle ϕ with QO, prove that

$$\frac{\text{vel. of } Q}{\text{vel. of } P}=\frac{\sin(\theta+\phi)}{\cos\phi}.$$

5. A load of weight W is being drawn along a rough horizontal surface by a rope inclined at an angle θ to the ground. Find by the Calculus the value of θ for which the tension is least, if λ is the angle of friction.

M. 21

1. A point moves so that its position at the end of t secs. is given by

$$x+2=\cos\frac{\pi t}{4}+\cos\frac{2\pi t}{4},$$

$$y=\sin\frac{\pi t}{4}.$$

Find its position, velocity and acceleration when $t=3$: x and y being in feet.

2. The jib of a crane is 20 ft. long and the vertical velocity of its end is 1 ft. per sec. What is the angular velocity of the rotation of the jib when it makes 30° with the vertical?

3. Fig. 144 represents a stage in a total solar eclipse, the Sun and Moon being of equal radius a. The shaded circle, centre O, represents the Moon moving at a uniform rate across the Sun whose centre is at O' and exactly covering it at the moment of totality. Find an expression in terms of θ radians for the fraction of area of the Sun which is uncovered. Calling the uncovered area A, find $\dfrac{dA}{d\theta}$. Show that $\dfrac{dA}{dr}$ = length of chord PQ where $OO'=r$. (Army.)

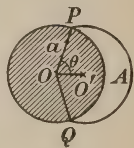

Fig. 144.

4. Prove that as x increases $\dfrac{a \sin x + b \cos x}{c \sin x + e \cos x}$ either increases for all values of x or else decreases for all values of x: find the condition to be satisfied by the constants a, b, c, e to provide that it shall always increase. (Cambridge University.)

5. A man is walking at 3 mls./hr. down a road 2 ft. from a wall BC. A road EF which turns off at right angles is 24 ft. wide. If A looks past the corner C at a wall EF on the opposite side of the other road, at what rate does this wall open up when AC makes an angle θ with CB? Prove that when $\theta=45°$ the acceleration of G is 232·32 ft./sec.^{2}

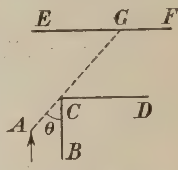

Fig. 145.

M. 22

1. The coordinates of a point are given by

$$x = 1 - \cos\theta; \quad y = \cos\theta + \tfrac{1}{4}\cos 2\theta:$$

prove that

$$\frac{dy}{dx} = -2\cos^2\frac{\theta}{2}.$$

2. A pole 30 ft. long is carried along a passage 12 ft. wide and into a corridor at right angles to the passage. Neglecting the thickness of the pole, find how wide the corridor must be in order that the pole may go round the corner without tilting one edge higher than the other.

3. If $\dfrac{\sin\phi}{\sin\phi'} = \mu\ (\mu > 1)$, prove

$$\frac{\tan\dfrac{\phi-\phi'}{2}}{\tan\dfrac{\phi+\phi'}{2}} = \frac{\mu-1}{\mu+1}.$$

Hence show that if ϕ increases then $\phi - \phi'$ also increases.

Also prove $\left(\dfrac{d\phi}{d\phi'}\right)^2 = \dfrac{\mu^2 - 1}{\cos^2 \phi} + 1$ and hence show that the increase of ϕ is greater than the increase of ϕ'.

4. A rod CD hinged at C is kept in contact with a door AB by a spring, so that as AB turns round A, D slides along BA. If the angular velocity of the door is ω, find the speed of D along BA when $\angle CDA = \theta$ and $AD = x$.

(Put $AC = d$ and $x = a \cos \theta + d \cos \phi$.)

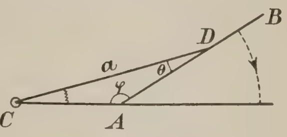

Fig. 146.

5. A rod AB of length $2l$ rests between two planes OP, OQ facing one another, whose inclinations are α and β. G, the centre of gravity of the rod, is at a distance c from A. If AB is inclined at θ to the horizontal, find the height of G above O and find for what value of θ it is a minimum.

M. 23

1. The time of oscillation of a pendulum is $t = 2\pi \sqrt{\dfrac{l}{g}}$ and g varies inversely as the square of the distance of the place from the centre of the earth. If the radius of the earth is taken as 4000 miles and a clock with a seconds pendulum ($t = 2$) is correct at sea-level, how much will it go wrong per day if carried to a height of $\frac{1}{10}$ of a mile?

2. A vertical wheel is revolving about its centre. If the radius is 4 ft., compare the vertical speed of a point on the rim with its horizontal speed when the abscissa of the point is 2 and the origin is at the centre.

3. A point moves in a circle of radius a with constant angular velocity ω. Show that after time t the coordinates of the point are $x = a \cos \omega t$ and $y = a \sin \omega t$ if $y = 0$ when $t = 0$.

Prove
$$\ddot{x} \cos \omega t + \ddot{y} \sin \omega t = -\omega^2 a$$
and
$$-\ddot{x} \sin \omega t + \ddot{y} \cos \omega t = 0.$$

Show that these equations prove that the acceleration of the point is toward the centre and that it is equal to $\omega^2 a$.

4. A variable quantity θ is given in terms of x by $\theta = a + x \sin\theta$ where a is a constant. Prove that when $x=0$,

$$\frac{d\theta}{dx} = \sin a \quad \text{and} \quad \frac{d^2\theta}{dx^2} = \sin 2a.$$

Show that an approximate value for θ when x is small is

$$a + x\sin a + \tfrac{1}{2}x^2 \sin 2a.$$

(Cambridge University.)

5. In a triangle ABC the side AB and the distance from C to the mid-point D of AB are accurately measured and found to be 60″ and 30·01″ respectively. The sides AC and BC are more roughly measured and found to be nearly equal. If CD makes an angle θ with CA, prove that if $\delta\theta$ is the amount θ differs from 45° then $\delta(\cot\theta) = {\cdot}000333$ and hence C is about 69″ less than a right angle.

CHAPTER XIII

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this chapter we are dealing with an entirely new kind of function. We are going to consider how to differentiate such a function as 10^x: the difference between this and the ordinary algebraic functions so far considered is that here the *variable occurs in the index.*

The formula $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ does not therefore apply to a function of this kind.

To find the value of $\frac{d}{dx}(10^x)$

We shall first of all prove that $\frac{d}{dx}(10^x)$ is equal to $10^x \times$ a numerical factor.

By definition,
$$\frac{d}{dx}(10^x) = \underset{h \to 0}{\text{Lt}} \frac{10^{x+h} - 10^x}{h}$$
$$= \underset{h \to 0}{\text{Lt}}\, 10^x \cdot \left(\frac{10^h - 1}{h}\right)$$
$$= 10^x \cdot \underset{h \to 0}{\text{Lt}} \left(\frac{10^h - 1}{h}\right).$$

Now if $\frac{10^h - 1}{h}$ tends to a limit when $h \to 0$, this limit obviously does not involve x and must be some constant number, call it c.

If the reader wishes to obtain an idea of the numerical value of c, he can do so by evaluating, using logarithm tables, the fraction $\frac{10^h - 1}{h}$ when h is small, say $h = 0{\cdot}01$, and he will then find that c is approximately $2{\cdot}3$. We shall however retain c for the present and employ a different method later for evaluating it.

Thus $\frac{d}{dx}(10^x) = c \cdot 10^x$, where c is a constant.

To find the value of $\dfrac{d}{dy}(\log_{10} y)$

Let $\log_{10} y = x$, $\therefore\ y = 10^x$.

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(10^x) = c \,.\, 10^x = c \,.\, y,$$

$$\therefore\ \frac{dx}{dy} = \frac{1}{c} \,.\, \frac{1}{y},$$

$$\therefore\ \frac{d}{dy}(\log_{10} y) = \frac{1}{c} \,.\, \frac{1}{y},$$

which gives us the required result involving the same constant c.

Reversing this equation we write

$$\frac{1}{c}\int \frac{dy}{y} = \log_{10} y + \text{a constant},$$

or

$$\int \frac{dy}{y} = c \log_{10} y + \text{a constant}.$$

We can now obtain the value of c (approximately) by taking a special case.

For we have $\displaystyle\int_1^2 \frac{dy}{y} = \Big[c \log_{10} y\Big]_1^2 = c \log_{10} 2 \triangleq c \times 0{\cdot}301.$

But we showed in Part I, p. 100, that $\displaystyle\int_1^2 \frac{dy}{y} \triangleq 0{\cdot}693.$

$$\therefore\ c \times 0{\cdot}301 \triangleq 0{\cdot}693,$$

$$\therefore\ c \triangleq \frac{0{\cdot}693}{0{\cdot}301} \triangleq 2{\cdot}30.$$

$\therefore$ we have

$$\frac{d}{dx}(10^x) = 2{\cdot}30 \times 10^x \quad\text{and}\quad \frac{d}{dx}(\log_{10} x) = \frac{1}{2{\cdot}30x},$$

where the numerical factor is approximate.

The presence of this numerical factor complicates these formulae and we therefore proceed to transform them so as to remove it. This transformation is analogous to that made in changing from degrees to circular measure in order to remove the numerical factor that occurs when we evaluate $\dfrac{d}{dx}\sin x$, etc., if x is measured in degrees.

From the formula $\frac{d}{dx}(10^x) = c \,.\, 10^x$ we have

$$\frac{d}{dx}\left(10^{\frac{x}{c}}\right) = \frac{d\left(10^{\frac{x}{c}}\right)}{d\left(\frac{x}{c}\right)} \times \frac{d\left(\frac{x}{c}\right)}{dx} = c \,.\, 10^{\frac{x}{c}} \times \frac{1}{c} = 10^{\frac{x}{c}}.$$

Now $$10^{\frac{x}{c}} = \left(10^{\frac{1}{c}}\right)^x.$$

If then we put $10^{\frac{1}{c}} = e$, we have $10^{\frac{x}{c}} = e^x$ and our formula becomes

$$\frac{d}{dx}(e^x) = e^x,$$

where $e = 10^{\frac{1}{c}} \doteq 10^{\frac{1}{2 \cdot 30}} \doteq 10^{\cdot 435} \doteq 2 \cdot 72$.

And further if we now use this number e as the base for our logarithms instead of 10, we remove the awkward numerical factor from the derivative of $\log y$.

Let $$\log_e y = x, \quad \therefore\ y = e^x.$$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x = y,$$

$$\therefore\ \frac{dx}{dy} = \frac{1}{y},$$

$$\therefore\ \frac{d}{dy}(\log_e y) = \frac{1}{y}.$$

The fact that the introduction of this number we have called e simplifies these two formulae, and the reasons why it does so, make the number e play a very important part in all higher mathematics: its value can be computed to as many places of decimals as is wished: to 9 places $e = 2 \cdot 718281828$: but just as in the case of π, this decimal never terminates nor recurs.

In future we shall use the symbol $\log x$ *to mean* $\log_e x$.

It should be noticed that the formula $\int \frac{dx}{x} = \log x$ supplies the

missing result in the general relation $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ which gives the integral of all powers of x except x^{-1}, but fails if $n=-1$.

Example 1.

Find (i) $\frac{d}{dx}(e^{ax})$; (ii) $\frac{d}{dx}\log(ax)$; (iii) $\frac{d}{dx}\log(\sin x)$.

(i) $$\frac{d}{dx}(e^{ax}) = \frac{de^{ax}}{d(ax)} \times \frac{d(ax)}{dx} = e^{ax} \times a = ae^{ax}.$$

(ii) $$\frac{d}{dx}\log(ax) = \frac{d\log(ax)}{d(ax)} \times \frac{d(ax)}{dx} = \frac{1}{ax} \times a = \frac{1}{x},$$

or better $$\frac{d}{dx}\log(ax) = \frac{d}{dx}(\log a + \log x) = \frac{1}{x}.$$

(iii) $$\frac{d}{dx}\log(\sin x) = \frac{d\log(\sin x)}{d(\sin x)} \times \frac{d(\sin x)}{dx} = \frac{1}{\sin x} \times \cos x = \cot x.$$

Example 2.

Find (i) $\int e^{ax}\,.\,dx$; (ii) $\int \frac{dx}{ax+b}$.

(i) Since $\frac{d}{dx}e^{ax} = a\,.\,e^{ax}$, we have $\int e^{ax}\,.\,dx = \frac{1}{a}\,.\,e^{ax} + c$.

(ii) Since $\frac{d}{dx}[\log(ax+b)] = \frac{1}{ax+b} \times a = \frac{a}{ax+b}$,

we have $$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b) + c,$$

or let $\int \frac{dx}{ax+b} = y$ so that $\frac{dy}{dx} = \frac{1}{ax+b}$; put $ax+b=z$;

$$\therefore \quad \frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = \frac{1}{ax+b} \times \frac{d}{dz}\left(\frac{z-b}{a}\right) = \frac{1}{z} \times \frac{1}{a};$$

$$\therefore \quad y = \frac{1}{a}\int \frac{dz}{z} = \frac{1}{a}\log z + c = \frac{1}{a}\log(ax+b) + c.$$

Note in particular $$\int \frac{dx}{b-x} = -\log(b-x) + c.$$

EXAMPLES XIII a

Differentiate with respect to x the expressions in Ex. 1—20 :

1. e^{2x}. **2.** $e^x + e^{-x}$. **3.** $e^x - e^{-x}$. **4.** e^{x^2}.

5. e^{ax+b}. **6.** $x\,e^x$. **7.** $e^{3x}\sin 2x$. **8.** $e^{-2x}\cos 3x$.

9. $\dfrac{e^{ax}}{\sin bx}$. **10.** $\log(3x)$. **11.** $\log(1-x)$. **12.** $\log(x^3)$.

13. $\log\cos x$. **14.** $\log\tan x$. **15.** $\log(x+\sin x)$.

16. $x^2\log x$. **17.** $\log\dfrac{a-x}{a+x}$. **18.** $\log\tan\dfrac{x}{2}$.

19. $\log(\tan x+\sec x)$. **20.** $\log\dfrac{1+\sin x}{1-\sin x}$.

Integrate with respect to x the expressions in Ex. 21—28:

21. e^{2x}. **22.** e^{-3x}. **23.** e^{ax+b}. **24.** $\dfrac{5}{x}$.

25. $\dfrac{3}{4-5x}$. **26.** $\tan x$. **27.** $\cot 3x$. **28.** $\dfrac{\cos x}{1+\sin x}$.

29. Show that $\dfrac{1}{x^2+5x+6}\equiv\dfrac{1}{x+2}-\dfrac{1}{x+3}$ and hence find $\displaystyle\int\frac{dx}{x^2+5x+6}$.

30. Find $\dfrac{d}{dx}(x\log x)$ and hence find $\displaystyle\int\log x\,.\,dx$.

31. Use anti-logarithm tables to evaluate $\dfrac{10^h-1}{h}$ when

$$h=0{\cdot}2,\ 0{\cdot}15,\ 0{\cdot}1,\ 0{\cdot}05,\ 0{\cdot}01,\ 0{\cdot}006.$$

32. If $y=\log_{10}x$ and if $\log_{10}e=0{\cdot}4343$, show that $\delta y \doteq \dfrac{0{\cdot}4343\,\delta x}{x}$; and given $\log_{10}5=0{\cdot}6990$, calculate $\log_{10}5{\cdot}1$.

33. If δy is the difference for $1'$ in a table of log-sines where $y=\log_{10}\sin x$, prove that $\delta y\doteq 0{\cdot}0001263\cot x$: $[\log_{10}e=0{\cdot}4343]$.

34. Prove that in a table of log-tangents to base 10 the difference of $1'$ in the neighbourhood of $60°$ is approximately $0{\cdot}00029$.

35. If $y=a\,.\,e^{2x}+b\,.\,e^{3x}$, prove that $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=0$.

36. Find the values of a if $y=e^{ax}$ satisfies the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0.$$

The Exponential Function

The relation $\dfrac{dy}{dx}=y$ which is satisfied by the function $y=e^x$, called the *exponential function*, is so important that it is necessary to discuss its meaning in detail. The relation states that y or e^x

is a function whose rate of increase per unit increase of x is equal to the function. This functional law has various physical applications:

(i) Consider for example the growth of bacilli in a culture: the rate of increase in the number of bacilli per unit of time is proportional to the number (N) present in the culture at that time (t), in symbols $\frac{dN}{dt} = k \cdot N$, where k is a constant.

(ii) Newton's law of cooling is of the same type: if the temperature of a body at any time exceeds that of the surrounding air (supposed constant) by $\theta°$, the rate of *decrease* of the temperature per unit increase of time (t) is proportional to θ, in symbols $\frac{d\theta}{dt} = -k\theta$, where k is a positive constant.

But the most familiar example is the increase of a sum of money at Compound Interest.

Suppose £1 invested at 100 % per annum comp. int. for x years, the interest being paid n times a year.

At the end of the first period,

$$£\,\frac{1}{n} \text{ interest is paid and the amount is } £\left(1+\frac{1}{n}\right).$$

The second period begins with $£\left(1+\frac{1}{n}\right)$,

$$\text{the interest for this period is } £\,\frac{1}{n}\left(1+\frac{1}{n}\right).$$

$\therefore$ the amount at the end of the second period

$$=\left(1+\frac{1}{n}\right)+\frac{1}{n}\left(1+\frac{1}{n}\right)=£\left(1+\frac{1}{n}\right)^2.$$

Similarly the amount at the end of the third period

$$=\left(1+\frac{1}{n}\right)^2+\frac{1}{n}\left(1+\frac{1}{n}\right)^2=£\left(1+\frac{1}{n}\right)^3.$$

And at the end of nx periods or x years, the amount

$$=£\left(1+\frac{1}{n}\right)^{nx}.$$

It should be noticed that the interest added on at the end of each period is a constant fraction $\left(\frac{1}{n}\right)$ of the sum accumulated at the beginning of the period.

The *Amount* therefore advances by a series of jumps which are not equal but increase successively, the rate of increase of the amount per period being $\frac{1}{n} \times$ the sum at the beginning of that period. Now keep the rate of interest fixed at 100 % per annum but suppose the interest paid at more frequent intervals, i.e. make n increase: the jumps become more numerous but each is smaller, and by taking n sufficiently large we can make each jump as small as we please. And in the limit ($n \to \infty$) we can regard the compound interest as being added on all the time (the money increasing like a snowball rolled in the snow) and we obtain a function whose rate of increase per unit time is proportional to its value at that time.

The Amount of £1 in x years at 100 % per annum paid continuously is therefore

$$\underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^{nx}.$$

But the law of growth of this function is precisely the law expressed by the relation $\frac{dy}{dx} = y$ which is satisfied by $y = e^x$, because the increase of the amount in $\frac{1}{n}$ th of a year is $\frac{1}{n}$ th of the amount at the beginning of that period and therefore the rate of increase *per unit time* (viz. one year) equals the amount at that moment.

Further, when $x = 0$, $e^x = e^0 = 1$ and the amount was originally £1.

$\therefore$ initially these two functions are equal.

But if two functions start by being equal and increase at the same rate as each other, they must always remain equal.

$\therefore$ £(e^x) is the amount to which £1 accumulates in x years at 100 % per annum continuous compound interest.

$$\therefore \underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^{nx} = e^x.$$

In particular, if $x=1$ we have $\operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$

Further it is clear that if the rate of interest is k hundred $^0/_0$ per annum continuous compound interest, the amount of £1 in x years would be $\operatorname*{Lt}_{n\to\infty}\left(1+\frac{k}{n}\right)^{nx}$.

But since $\frac{d}{dx}(e^{kx})=k\,.\,e^{kx}$, this is the rate of increase which e^{kx} follows: also when $x=0$, $e^{kx}=e^0=1$.

$$\therefore \operatorname*{Lt}_{n\to\infty}\left(1+\frac{k}{n}\right)^{nx}=e^{kx},$$

and in particular putting $x=1$ we have $\operatorname*{Lt}_{n\to\infty}\left(1+\frac{k}{n}\right)^{n}=e^k.$

The law embodied in the equation

$$\frac{dy}{dx}=ky,$$

where k is a constant which is satisfied by $y=e^{kx}$, is therefore called the *Compound Interest Law*: and any function which obeys it is called a Growth Function (e.g. the growth of the bacilli, p. 190).

If we start with £P, the amount at the end of x years at k hundred $^0/_0$ per annum continuous compound interest will be £$P\,.\,e^{kx}$: this obeys the same law.

$\therefore$ $y=P\,.\,e^{kx}$, *where P is an arbitrary constant, is the general solution of the relation* $\frac{dy}{dx}=ky$.

This result could be obtained as follows:

$$\frac{dy}{dx}=ky, \quad \therefore \frac{dx}{dy}=\frac{1}{ky} \quad \text{or} \quad x=\frac{1}{k}\int\frac{dy}{y}=\frac{1}{k}\log y+c.$$

$\therefore$ $\log y=k(x-c)$ or $y=e^{kx-kc}=b\,.\,e^{kx}$, where b is a constant.

But when $x=0$, $y=P$; $\quad\therefore P=b.$

$$\therefore y=P\,.\,e^{kx}.$$

The above discussion shows us the rough shape we may expect the graph of e^x to take.

At $x=0$, $e^x=e^0=1$ and $\dfrac{d}{dx}(e^x)=1$.

$\therefore$ the slope of the curve at this point is 45° upwards.

Also the greater e^x becomes, the more rapidly it increases.

When x is negative and equal to $-z$ say, $e^{-z}=\dfrac{1}{e^z}$ and is $\therefore$ positive and less than 1.

Its shape is therefore as shown in Fig. 147.

A table of logarithms to base e and a table giving the values of powers of e will be found at the end of this book. By making use of the latter, the graph of e^x can be quickly drawn.

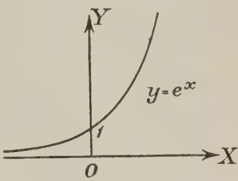

Fig. 147.

EXAMPLES XIII b

1. Plot the graph of $y=e^x$ from $x=-1$ to $x=2$ and then draw the reflection of this curve in the line $y=x$. If the equation of the new curve is $y=f(x)$, what is $f(x)$?

2. Find y if $\dfrac{dy}{dx}=2y$ and $y=3$ when $x=0$.

3. Find y if $\dfrac{dy}{dx}+3y=0$ and $y=2$ when $x=0$.

4. Sketch the graph of $y=e^x$, mark the points $P(x, y)$, $Q(x+h, y+k)$ on it, also the points $P'(x', y')$, $Q'(x'+h, y'+k)$; prove that

$$\frac{\text{gradient of chord } PQ}{y}=\frac{\text{gradient of chord } P'Q'}{y'}.$$

What does this become when $h \to 0$?

5. What is the area of the figure bounded by the x-axis, the ordinates $x=0$, $x=1$ and the curve $y=e^x$?

6. What is the area of the figure bounded by the x-axis, the y-axis and the curve $y=e^{-x}$?

7. Draw the graph of $y=e^{-x^2}$. [The curve of normal error.]

8. An electric current C is decreasing according to the law $C=ke^{-\frac{t}{\lambda}}$ where k, λ are constants; compare the amount of electricity $\int C\,dt$ passing in time T with the amount passing altogether.

9. The speed of signalling in a submarine telegraph cable is proportional to $x^2 \log\left(\frac{1}{x}\right)$, where x is the ratio of the radius of the core of copper wire to the thickness of the covering. Show that for a maximum speed $x = \frac{1}{\sqrt{e}}$.

10. The amount x of a substance undergoing transformation in a chemical reaction at time t is given by $x = ae^{-kt}$, where a is the amount present at the beginning: show that the velocity of the reaction is proportional to the amount of substance undergoing transformation.

11. *Sketch* the graph of $y = e^{-\frac{x}{2}} \sin x$.
Calculate its gradient when $x = 0$ and when $x = \pi$.

12. Find the length of the subtangent at any point of the curve (i) $y = e^x$, (ii) $y = a \, . \, e^{\frac{x}{b}}$.

13. A pane of glass absorbs 2 per cent. of the light incident upon it. How much light will pass through 12 panes of this glass?

14. Does $\log x$ increase more or less rapidly than x as x increases from 0·1 to 10?

15. A particle moves so that when its velocity is v ft./sec. its acceleration is $\frac{1}{5}v$ ft./sec.^{2} Its initial velocity is 5 ft./sec., find its velocity after 10 secs.

16. If the tangent at $P\,(x, y)$ on the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ makes an angle ψ with the x-axis, prove that $y \cos \psi = a$: and if the tangent cuts the y-axis at T, prove that $PT = \frac{xy}{a}$.

To find $\frac{d}{dx}(a^x)$

Let $a^x = y$, $\therefore \log y = \log(a^x) = x \log a$.

$$\therefore \frac{d}{dx}(\log y) = \log a \frac{d}{dx}(x) = \log a,$$

but
$$\frac{d}{dx}(\log y) = \frac{d(\log y)}{dy} \times \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx},$$

$$\therefore \frac{dy}{dx} = y \log a.$$

$$\therefore \ \frac{d}{dx}(a^x) = a^x \log a,$$

where $\log a \equiv \log_e a$.

Note. When the index is variable, it is often advisable as here to take logarithms before differentiation.

To find $\frac{d}{dx}(\log_a x)$

Let $\log_a x = y$, $\therefore \ x = a^y$.

$$\therefore \ \frac{d}{dx}(x) = \frac{d}{dx}(a^y) = \frac{d(a^y)}{dy} \times \frac{dy}{dx} = a^y \log a \frac{dy}{dx},$$

$$\therefore \ a^y \log a \frac{dy}{dx} = 1.$$

$$\therefore \ \frac{d}{dx}(\log_a x) = \frac{dy}{dx} = \frac{1}{a^y \log a} = \frac{1}{x \log a}.$$

This result can be obtained more quickly by using the identity

$$\log_a x \equiv \frac{\log_e x}{\log_e a}.$$

When differentiating expressions consisting of two or more factors, it is sometimes helpful to start by taking logarithms: this is illustrated in the next example.

Example 3.

If the pressure p and the volume v of a gas obey the adiabatic law $pv^\gamma = \text{const.}$ where $\gamma \triangleq 1{\cdot}414$, find the percentage decrease in p corresponding to a small increase x per cent. in v.

$$pv^\gamma = c, \quad \therefore \ \log p + \log(v^\gamma) = \log c,$$

or

$$\log p + \gamma \log v = \log c.$$

$$\therefore \ \frac{1}{p}\frac{dp}{dv} + \frac{\gamma}{v} = 0 \ \text{ or } \ \frac{\delta p}{p} \triangleq -\gamma \cdot \frac{\delta v}{v}.$$

But

$$\delta v = \frac{vx}{100}, \quad \therefore \ \frac{\delta p}{p} \triangleq -\frac{\gamma x}{100}.$$

$\therefore$ the decrease in p is approximately γx per cent.

Example 4.

The differential equation for damped vibrations of a pendulum is

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + (k^2 + \omega^2)e^{kt} = 0.$$

Show that this is satisfied by $\theta = Ae^{-kt}\sin(\omega t+a)$, where A, a, k, ω are constants.

We have $$\theta \,.\, e^{kt} = A\sin(\omega t+a).$$

Differentiate with respect to t.

$$\therefore\; e^{kt}\frac{d\theta}{dt} + k\theta \,.\, e^{kt} = A\omega\cos(\omega t+a).$$

Differentiate again.

$$\therefore\; e^{kt}\frac{d^2\theta}{dt^2} + k \,.\, e^{kt}\frac{d\theta}{dt} + k\frac{d\theta}{dt} \,.\, e^{kt} + k^2\theta \,.\, e^{kt} = -A\omega^2\sin(\omega t+a),$$

$$\therefore\; e^{kt}\left(\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + k^2\theta\right) = -\omega^2\theta \,.\, e^{kt},$$

$$\therefore\; \frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + (k^2+\omega^2)\theta = 0.$$

EXAMPLES XIII c

1. Differentiate with respect to x:

(i) 10^x, (ii) $\log_{10}(2x)$, (iii) $\dfrac{1}{3^x}$,

(iv) $\log_{10}(1+x^2)$, (v) $\log_{10}(\cos x)$, (vi) $2^x\log_{10} x$.

2. Integrate with respect to x:

(i) 10^x, (ii) 3^{2x}, (iii) a^{cx}.

3. Evaluate to two significant figures:

(i) $\displaystyle\int_2^3 \frac{dx}{x}$, (ii) $\displaystyle\int_1^2 10^x\,dx$, (iii) $\displaystyle\int_0^1 e^{2x}\,dx$.

4. If $2u = e^x - e^{-x}$ and $2v = e^x + e^{-x}$, prove that

(i) $\dfrac{du}{dx} = v$, (ii) $\dfrac{dv}{dx} = u$, (iii) $\dfrac{d}{dx}(uv) = u^2 + v^2$.

5. If $(1+x)y = e^x$, prove that $(1+x)\dfrac{dy}{dx} = xy$.

6. Find $\dfrac{d^3y}{dx^3}$ if (i) $y = a \,.\, e^{cx}$, (ii) $y = \log(bx)$.

7. (i) If $pv^\gamma = c$, prove that $v\dfrac{dp}{dv} + \gamma p = 0$.

(ii) If $y = (x-a)^p(x-b)^q(x-c)^r$, prove that

$$\frac{1}{y}\frac{dy}{dx} = \frac{p}{x-a} + \frac{q}{x-b} + \frac{r}{x-c}.$$

8. If $pv=c$, prove that $\int_{v_1}^{kv_1} p\,dv = c\log k$.

9. Prove that $y=e^{2x}(ax+b)$ satisfies the equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0.$$

10. Prove that $t=\frac{1}{2ku}\log\left(\frac{u+v}{u-v}\right)$ satisfies the equation

$$\frac{dv}{dt}=k(u^2-v^2),$$

where k, u are constants.

11. Assuming that $\int e^{ax}\sin bx\,dx=e^{ax}(p\sin bx+q\cos bx)$ where p, q are constants, express p and q in terms of a, b by differentiating and equating coefficients.

12. The equation of the catenary is $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$; if

$$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},$$

and if $s=0$ when $x=0$, express s in terms of x and prove that $y^2=c^2+s^2$.

13. Pressure is brought to bear on a circular plate of radius 1 in.; the pressure at any point distant x ins. from the centre is κe^{-x} tons per sq. in. and the maximum pressure is 40 tons per sq. in. Find the average pressure on the plate in tons per sq. in.

14. For a certain curve the ordinates corresponding to $x=0, 1, 2, 3, 4, 5, 6$ are in geometrical progression of ratio 1·2. If $y=1$ when $x=0$, plot the curve. Show that the curve is $y=1{\cdot}2^x$ and express this in the form $y=e^{bx}$.

Find $\frac{dy}{dx}$ when $x=4$ and evaluate $\int_0^4 y\,dx$. (Army.)

15. Prove that $\log x+\frac{1}{x}$ is not less than 1 for any positive value of x.

16. If the deflection of the needle of a damped galvanometer is given by $\theta=Ce^{-\kappa t}\sin\left(\frac{\kappa\pi}{\lambda}t\right)$, prove that

(i) The maximum deflection is when $t=\frac{\lambda}{\kappa\pi}\tan^{-1}\left(\frac{\pi}{\lambda}\right)$;

(ii) If Ω is the initial angular velocity, $C=\frac{\Omega\lambda}{\kappa\pi}$.

We shall now indicate some of the numerous physical applications of the Compound Interest Law.

(i) *Newton's Law of Cooling.*

If the temperature of a body is $\theta°$ (Centigrade) above the temperature of the surrounding air which remains constant, the rate of decrease of θ is proportional to θ.

Example 5.

A body in a room of temperature 15° C. starts at a temperature of 75° C. and ten minutes later its temperature is 55° C. What is its temperature after a further five minutes?

Let the temperature be $\theta°$ C. above that of the room after t minutes.

Then we have $\dfrac{d\theta}{dt} = -k\theta$, where k is a constant.

$$\therefore\ \theta = \theta_0\, e^{-kt}.$$

But when $t=0$, $\theta = 75-15=60$, $\therefore\ \theta_0 = 60$.

$$\therefore\ \theta = 60e^{-kt}.$$

But when $t=10$, $\theta = 55-15=40$, $\therefore\ 40 = 60e^{-10k}$.

$$\therefore\ e^{-10k} = \frac{40}{60} = \frac{2}{3},\ \therefore\ e^{-k} = \left(\frac{2}{3}\right)^{\frac{1}{10}}.$$

[We could of course find k from this equation, but it is unnecessary.]

$\therefore$ when $t=15$, $\theta = 60e^{-15k} = 60\left[\left(\dfrac{2}{3}\right)^{\frac{1}{10}}\right]^{15} = 60\left(\dfrac{2}{3}\right)^{\frac{15}{10}}.$

$$\therefore\ \theta = 60\left(\frac{2}{3}\right)^{1\cdot 5} = 32\cdot 7.$$

$\therefore$ the temperature $= 15 + 32\cdot 7 = 47\cdot 7°$ C.

(ii) *Coefficient of Expansion.*

If a body is heated, it expands so that the fractional rate of increase of volume per unit increase in temperature is approximately constant; the constant is called the coefficient of cubical expansion.

If the volume is V cu. cms. when the temperature is $\theta°$ C. and the coefficient of expansion is k, we have

$$\frac{\dfrac{dV}{d\theta}}{V} = k \quad\text{or}\quad \frac{dV}{d\theta} = kV.$$

$$\therefore\ V = V_0 \cdot e^{k\theta},$$

where V_0 is the volume at 0° C.

(iii) *Atmospheric Pressure* in still air at constant temperature.

Suppose the pressure at a height of x feet above the ground is p lbs. per sq. in. and that 1 cu. ft. of air at pressure p weighs w lbs.

Consider a column of air on a base of area α sq. ins. between two horizontal planes at heights x and $x+\delta x$ feet.

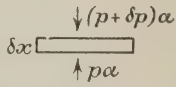

Fig. 148.

The weight of this column is $w\dfrac{\alpha}{144}\delta x$ lbs. But the net supporting force is

$$p\alpha-(p+\delta p)\alpha=-\alpha\delta p \text{ lbs.}$$

$$\therefore\ -\alpha\delta p=w\frac{\alpha}{144}\delta x,$$

or

$$\frac{dp}{dx}=-\frac{w}{144}.$$

Now the volume v of a given mass of gas $\propto \dfrac{1}{w}$, where w is the density. But by Boyle's law

$$v\propto\frac{1}{p}.$$

$$\therefore\ w\propto p,$$

$$\therefore\ \frac{w}{p}=\text{const.}=\lambda \text{ say},$$

$$\therefore\ \frac{dp}{dx}=-\frac{\lambda}{144}p.$$

$$\therefore\ p=p_0e^{-\frac{\lambda x}{144}},$$

where $p=p_0$ when $x=0$, i.e. on the ground.

If 1 cu. ft. of air at the ground weighs w_0 lbs., we have $\lambda=\dfrac{w_0}{p_0}$ and our equation becomes

$$\frac{p}{p_0}=e^{-\frac{w_0x}{144p_0}}.$$

(iv) *Leakage in an electric condenser.*

If a condenser contains a charge, the leakage at any time is proportional to the charge.

If the initial charge is Q_0 and if after t secs. the charge falls to Q, we have

$$\frac{dQ}{dt} = -\lambda Q, \text{ where } \lambda \text{ is a constant.}$$

$$\therefore\ Q = Q_0 \cdot e^{-\lambda t}.$$

(v) *Tension of a belt passing over a rough circular pulley.*

Consider a small portion PQ of the belt which subtends an angle $\delta\psi$ radians at the centre O of the pulley. The belt is on the point of sliding in the direction $P \rightarrow Q$. The angle between the tangents PKH and KQ is $\delta\psi$. We can replace $T + \delta T$ along KQ by $(T + \delta T) \sin \delta\psi$ perpendicular to KH and $(T + \delta T) \cos \delta\psi$ along KH.

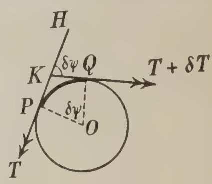

Fig. 149.

$\therefore$ for small values of $\delta\psi$ the effect of the tensions on PQ is represented by

(i) along PH, $(T + \delta T) \cos \delta\psi - T \doteqdot \delta T$ since $\cos \delta\psi \doteqdot 1$,

(ii) perpendicular to PH, $(T + \delta T) \sin \delta\psi \doteqdot T\delta\psi + \delta T \delta\psi \doteqdot T\delta\psi$.

But this is balanced by the normal reaction R and the friction μR, if μ is the coefficient of friction.

$$\therefore\ T\delta\psi \doteqdot R \text{ and } \delta T \doteqdot \mu R,$$

$$\therefore\ \delta T \doteqdot \mu T \delta\psi,$$

$$\therefore\ \frac{dT}{d\psi} = \mu T.$$

$$\therefore\ T = T_0 \cdot e^{\mu\psi},$$

where T_0 is the tension when $\psi = 0$.

Example 6.

A cable makes one complete turn round a circular post, the coefficient of friction being $\frac{3}{4}$. One end is held by a man who can exert a force of 40 lbs.; what strain can the other end support?

Here $\mu = \frac{3}{4}$, $T_0 = 40$, $\psi = 2\pi$ (one complete turn).

$$\therefore\ T = 40 e^{\frac{3\pi}{2}}$$

$$= 4440 \text{ lbs. or nearly 2 tons.}$$

Example 7.

A vertical column 15 feet high has a plane horizontal top of area 1·5 sq. ins. which carries a weight of 30 lbs. The column is so shaped that the pressure per sq. in. across every horizontal section is constant for all sections. If the material weighs $\frac{1}{4}$ lb. per cu. in., find the area of the base. If the column is a solid of revolution, what curve must be used to generate it?

The pressure on the top is 30 lbs. per 1·5 sq. ins. which equals 20 lbs. per sq. in.

$\therefore$ the pressure across every horizontal section must be 20 lbs. per sq. in.

Let the area of a section x ins. from the top be A sq. ins. Then the forces on the portion between two sections at distances x ins. and $x+\delta x$ ins. from the top are shown in Fig. 150. For the support from below is $20(A+\delta A)$ lbs. and the pressure from above is $20A$ lbs. and the weight of the section (volume $A\,\delta x$ cu. ins.) is $\frac{1}{4}A\,\delta x$ lbs.

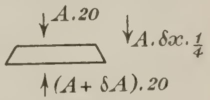

Fig. 150.

[Note the weight really lies between $\frac{1}{4}A\,\delta x$ and $\frac{1}{4}(A+\delta A)\,\delta x$, but the product $\delta A\,\delta x$ disappears in the limit.]

$\therefore$ the net support $=20(A+\delta A)-20A=20\delta A$ lbs.

$$\therefore\ 20\delta A \bumpeq \tfrac{1}{4}A\,\delta x \quad\text{or}\quad \frac{\delta A}{\delta x} \bumpeq \frac{A}{80}.$$

$\therefore$ in the limit $\dfrac{dA}{dx}=\dfrac{A}{80}$, $\therefore\ A=A_0\,e^{\frac{x}{80}}$.

But when $x=0$, $A=1{\cdot}5$. $\therefore\ A_0=1{\cdot}5$.

$$\therefore\ A=1{\cdot}5e^{\frac{x}{80}}.$$

Now the height $=15$ feet $=180$ ins.

$$\therefore\ \text{area of base}=1{\cdot}5e^{\frac{180}{80}}=1{\cdot}5e^{2\cdot25}\ \text{sq. ins.}$$
$$=1{\cdot}5\times 9{\cdot}49=14{\cdot}2\ \text{sq. ins.}$$

Suppose the radius of the circular section at depth x ins. is y ins., then

$$\pi y^2=1{\cdot}5\,e^{\frac{}{80}}.$$

$$\therefore\ y=\sqrt{\frac{1{\cdot}5}{\pi}}\,e^{\frac{x}{160}}.$$

$$\therefore\ y=0{\cdot}69\,e^{\frac{x}{160}}.$$

If this curve is revolved about the x-axis, the axis of the column, the required solid will be generated.

EXAMPLES XIII d

1. A body starts with velocity u ft./sec. and moves with an acceleration kv ft./sec.^{2} where v is its speed: find the distance it travels in t secs.

2. The temperature of a liquid in a room of constant temperature 20° C. is observed to be 70° C. After 5 minutes it is 60°. What will it be after another 30 minutes?

3. The current C at time t in a conductor is falling off according to the law $\frac{dC}{dt}+kC=0$, where k is a constant: initially $C=2$ and after $\frac{1}{10}$ sec. $C=1$; find k and the time taken for the current to fall to 0·01.

4. When a shell is moving with velocity v ft./sec., the air resistance causes a retardation $\frac{1}{10}(v-500)$ ft./sec.^{2} as long as $v>1000$. Its muzzle velocity is 2500 ft./sec.: what is its velocity after 5 seconds?

5. A tank which starts full is being emptied so that the rate at which the water runs out is proportional to the amount left in. If half the water runs out in 10 minutes, what fraction will remain after a quarter of an hour from the start?

6. A body of volume 1000 cu. cms. is at temperature 20° C.; when the temperature rises 10° C. the volume increases by 1 cu. cm. What is its volume at 100° C.?

7. For a condenser of capacity K charged with a quantity Q of electricity at potential V, if R is the resistance in the circuit, we have

$$Q=KV,\ \ V=RC,\ \ C=-\frac{dQ}{dt}.$$

If $V=V_0$ when $t=0$, prove that $R=t\div\left\{K\log\frac{V_0}{V}\right\}$.

8. A bullet is shot vertically upwards with velocity u ft./sec.; owing to air resistance its retardation is $g(1+kv^2)$ ft./sec.^{2} when its velocity is v ft./sec. Prove that the greatest height it reaches is $\frac{1}{2gk}\log(1+ku^2)$ feet.

9. A long metal tube, internal radius 1 cm. and external radius 2 cms., is filled with steam kept at temperature 125° C., the temperature in the metal at r cms. from the axis of the tube is T° C. where $\frac{dT}{dr}=-\frac{100}{r}$; find the temperature at the outer surface.

10. How many turns must be taken round a rough circular post, coefficient of friction $\frac{1}{4}$, so that a pull of 2 tons can be resisted by a force of 60 lbs.?

11. A rotating flywheel is subject to a frictional couple proportional to its angular velocity: its initial velocity is ω_1, and after 1 second it is ω_2; find its angular velocity after t seconds.

12. A jar of water at 15° C. is placed in a temperature of -12° C. and its temperature falls 5° in 8 minutes. How long will it be before ice begins to form?

13. A raindrop falls with acceleration $g-\frac{v}{2}$ ft./sec.^{2}, where v ft./sec. is its velocity. What is its limiting velocity and how far does it fall in 10 secs. from rest?

14. If the vapour pressure P and the absolute temperature T are connected by the equation $P=aT^n e^{\frac{b}{T}}$, where a, b, n are constants, find the value of T for which P is stationary. What is the condition that this value of P is a maximum?

15. One end of a metal bar in air at temperature 0° C. is kept a constant temperature T° C.; heat flows along the bar so that at a distance x feet from that end the temperature θ° is given by $\frac{d^2\theta}{dx^2}=n^2\theta$, where n is a constant: show that $\theta=Ae^{nx}+Be^{-nx}$ and use the fact that if the bar is very long the temperature at the far end remains at 0° C. to find A, B.

16. In a chemical change the rate of decomposition of a substance is proportional to the amount C that remains at any time t, and the percentage increase of pressure p is proportional to the percentage decrease of the substance: the initial values of C, p are C_0, p_0; express C and p in terms of t and the constants of variation, and find a relation between p, C.

17. A long vertical rod carries a weight of 5000 lbs. at its lower end; each horizontal section of the rod is of such a size that the stress across every section is 2000 lbs. per sq. in. and the material weighs $\frac{1}{4}$ lb. per cu. in. What is the area of the cross-section (i) at the lower end, (ii) 50 feet above the lower end, (iii) 200 feet above the lower end?

18. A rocket of weight 10 lbs. is fired vertically upwards starting from rest; the effect of the gradual burning of the charge is to produce an acceleration of $\frac{15g}{10-t}$ ft./sec.^{2} after t secs. The charge is consumed after 5 secs.; what is then the rocket's velocity? Take $g=32$.

19. When the voltage in a circuit of self-induction L suddenly changes from 0 to V, the current C after time t is given by

$$V=RC+L\frac{dC}{dt};$$

prove that

$$C=\frac{V}{R}+Ae^{-\frac{Rt}{L}},$$

and find A if $C=0$ when $t=0$.

20. A body A of mass 40 lbs. lies on a rough horizontal plane AC (coefficient of friction $\frac{1}{4}$) and is pulled slowly along the plane by a rope passing over a smooth pulley B: BC is vertical and equals 5 feet. When $AC=x$ feet, the work δW ft.-lbs. done in a small displacement δx ft. is given by

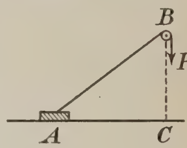

Fig. 151.

$$\delta W \doteq -10\delta x \Big/ \left(1+\frac{5}{4x}\right).$$

Find the work done if the initial and final lengths of AC are 20 feet and 5 feet.

CHAPTER XIV

DIFFERENTIALS AND GENERAL INTEGRATION

Differentials

LET the tangent at any point P of the curve $y=f(x)$ cut at S the ordinate of another point Q on the curve. Then $NM=PR$ being δx, RQ will be δy. The tangent of the angle SPR will be $\frac{dy}{dx}$ or $f'(x)$; hence $\frac{RS}{PR}=f'(x)$, i.e. $RS=f'(x)\,PR$. The increments RS and PR are known as *differentials* and are called dy and dx respectively.

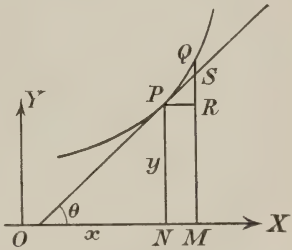

Fig. 152.

Hence $dy=f'(x)\,dx$ or $\frac{dy}{dx}dx$; dy and dx are therefore two quantities such that the ratio between them is the differential coefficient $\frac{dy}{dx}$.

Now dx and δx are quite arbitrary but when δx and dx are fixed it is possible to calculate δy and dy.

When $dx=\delta x=NM$, then dy is RS but δy is RQ; $\therefore$ δy is not equal to dy, but the ratio $\frac{dy}{\delta y}$ can be made as nearly equal to 1 as we please by taking δx sufficiently small.

The most useful application of differentials arises in the methods of Integration which will be discussed in the next section.

Product and Quotient Formulae in terms of Differentials

We have shown that $\frac{d\,(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$.

Now $du = \frac{du}{dx}\,dx$, $dv = \frac{dv}{dx}\,dx$, and $d\,(uv) = \frac{d}{dx}(uv)\,dx$;

$$\therefore\ d\,(uv) = u\,dv + v\,du.$$

Similarly $$\frac{d\,(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}.$$

This result may be most easily obtained by using logarithms, thus:

$$\log\,(uvw) = \log u + \log v + \log w,$$

$$\therefore\ \frac{d}{dx}\log\,(uvw) = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w;$$

i.e. $$\frac{1}{uvw}\,\frac{d\,(uvw)}{dx} = \frac{1}{u}\,\frac{du}{dx} + \frac{1}{v}\,\frac{dv}{dx} + \frac{1}{w}\,\frac{dw}{dx},$$

or $$\frac{d\,(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}.$$

Similarly $$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}.$$

$$\therefore\ d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

In other words the relation between Differentials is obtained by treating $\frac{dy}{dx}$ as if it were a fraction; thus if $y = x^2$ then $\frac{dy}{dx} = 2x$ and $dy = 2x\,dx$.

Example 1.

If $y = 3\,(2x+1)^2$, find dy.

$$\frac{dy}{dx} = 12\,(2x+1),\quad \therefore\ dy = 12\,(2x+1)\,dx;$$

or $$d\,[3\,(2x+1)^2] = 12\,(2x+1)\,dx.$$

Example 2.

Find the differential of $\sin\theta$.

Since $$\frac{d\,(\sin\theta)}{d\theta} = \cos\theta,\quad \therefore\ d\,(\sin\theta) = \cos\theta\,.\,d\theta.$$

Application of Differentials

Errors. Since δy is approximately equal to the differential dy when δx is small, the calculation of the effect of errors may conveniently be carried out by using differentials.

Example 3.

The hypotenuse of a triangle is calculated from $c^2=a^2+b^2$ when $a=8{\cdot}21$ cms., $b=4{\cdot}03$ cms. measured to two places of decimals.

Find the greatest possible error in c.

Since $$c^2=a^2+b^2 \text{ and } d(c^2)=2c\,dc,$$

we have $$2c\,dc=2a\,da+2b\,db,$$

i.e. $$dc=\frac{a}{c}\,da+\frac{b}{c}\,db.$$

By calculation $c=9{\cdot}145$, also da and db may both be ${\cdot}005$;

$$\therefore\ dc=\frac{8{\cdot}21\times{\cdot}005+4{\cdot}03\times{\cdot}005}{9{\cdot}145}=\frac{12{\cdot}24\times{\cdot}005}{9{\cdot}145}=0{\cdot}0067.$$

It will be found convenient to use Differentials when dealing with more than two variables.

Example 4.

Find the least area of canvas that can be used to construct a conical tent whose cubical capacity is 800 cu. ft.

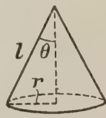

Fig. 153.

We have $$V=800=\tfrac{1}{3}\pi r^3\cot\theta \quad \text{..................(i)},$$

and A the area of canvas $=\pi rl$

$$=\pi r^2\operatorname{cosec}\theta \quad \text{...............(ii)}.$$

Instead of expressing A in terms of r or θ from (i) proceed as follows:

From (i) $r^3\cot\theta=$const., $\therefore\ d(r^3\cot\theta)=0$,

i.e. $$r^3(-\operatorname{cosec}^2\theta\,d\theta)+\cot\theta\,(3r^2dr)=0 \quad \text{...............(iii)}.$$

From (ii) since A is to be a minimum $dA=0$,

i.e. $$\pi\,[r^2(-\operatorname{cosec}\theta\cot\theta\,d\theta)+\operatorname{cosec}\theta\,2r\,dr]=0 \text{............(iv)}.$$

Now eliminate $d\theta$ and dr from (iii) and (iv);

$$\frac{d\theta}{dr}=\frac{3r^2\cot\theta}{r^3\operatorname{cosec}^2\theta}=\frac{2r\operatorname{cosec}\theta}{r^2\operatorname{cosec}\theta\cot\theta},$$

$$\therefore\ 3\cot^2\theta=2\operatorname{cosec}^2\theta=2+2\cot^2\theta,$$

$$\therefore\ \cot^2\theta=2,\quad \theta=35^\circ\,15',\quad r=8{\cdot}14,\quad \underline{A=361 \text{ sq. ft.}}$$

EXAMPLES XIV a

1. Write down the differentials of the following:

(i) ax^2+b; (ii) a^2-2x^2; (iii) $3x^2-5x+2$;
(iv) x; (v) $x^{\frac{1}{2}}$; (vi) $x^{-\frac{1}{2}}$;
(vii) $x+\frac{1}{x}$; (viii) $\cos\theta$; (ix) $\sec\theta$.
(x) $\tan(2x+3)$; (xi) $\tan^{-1}x$; (xii) $\sec^2\theta$;
(xiii) $\sin(3x+5)$; (xiv) $\operatorname{cosec}(2a+x)$; (xv) $\sec(a-x)$;
(xvi) $\log x$; (xvii) $\log(ax+b)$; (xviii) $x^{\frac{1}{2}}+x^{-\frac{3}{2}}$;
(xix) e^{ax}; (xx) e^{-3x}.

2. Write down the functions of which the following are the differentials:

(i) dx; (ii) $(2x+3)\,dx$; (iii) $x^{-\frac{1}{2}}\,dx$;
(iv) $\frac{dv}{v^{1\cdot 4}}$; (v) $\frac{du}{u^{\frac{3}{2}}}$; (vi) $\frac{du}{u+1}$;
(vii) $\frac{x^2-3}{x^2}\,dx$; (viii) $\sec\theta\tan\theta\,d\theta$; (ix) $\operatorname{cosec}^2\theta\,d\theta$.

3. Find the error in the area of a triangle calculated from $\frac{1}{2}ab\sin C$ when $a=6\cdot 4$, $b=5\cdot 8$ and $C=36°\,24'$ if there is an error of $2'$ in C.

4. An error of 2 °/ₒ is made in measuring an angle θ as $48\cdot 4°$. Find the percentage error in calculating the value of $\sin\theta+\cos\theta$.

5. A beam supported at its ends, and loaded at the mid-point with a weight W, sags through a distance d given by $d=WK\frac{l^3}{bt^3}$, where K is constant.

If $l=10$ ft., $b=6$ ins., $t=8$ ins. and each measurement may be ·1 °/ₒ in error, find the possible percentage error in the deflection. (Take logs first.)

6. The weight of copper (W) deposited in t secs. by a current C passing through copper sulphate is $W=Cte$, where e is the electro-chemical equivalent of copper. If the current can be read to a °/ₒ, the weight to b °/ₒ, the time to c °/ₒ, find the possible percentage error in calculating e.

7. A long rectangular strip of paper has a width AB of 4 ins. One corner A is folded over so that A comes on the long edge through B at A'. If the crease is CD where C is on AB, prove that $x(\cos 2\theta+1)=4$ where $x=AC$ and $\theta=\angle ADC$. Hence find the smallest area of the ΔACD.

8. Find the shape of the cylinder with the greatest volume for a given surface area A.

9. In using the formula $c^2=a^2+b^2-2ab\cos C$ for finding c, the errors in a and b are both $+2$ °/$_\circ$, but C is correct. Find the resultant percentage error in c.

10. Find the possible error in C if the formula for $\cos C$ is used when $a=64$, $b=27$, $c=43$ and there may be an error either way of 1 °/$_\circ$ in each measurement.

11. A crank $CP\,(r)$ rotates about C while the connecting rod $PQ\,(l)$ moves so that Q travels along a line QC. If, when CP makes an angle θ with CQ and PQ makes an angle ϕ with QC, then the angular velocity of QP is ω' and the angular velocity of CP is ω; prove that the velocity of Q is $-l\omega'\sin\phi-r\omega\sin\theta$.

Also prove $\dfrac{\omega'}{\omega}=\dfrac{r\cos\theta}{l\cos\phi}$ and $\dfrac{\text{vel. of } Q}{\text{vel. of } P}=\dfrac{\sin(\theta+\phi)}{\cos\phi}$.

12. A is a fixed point 1 foot above a fixed horizontal line MN. A rod AB, 6 ins. long, swings below A through $\theta°$ on the left of the vertical and an arm BC 2 feet long moves so that C slides along MN to the right of the vertical. Find the velocity of C when $\theta=15°$, if the velocity of B is then 4 ins. per sec.

13. Given $t=2\pi\sqrt{\dfrac{l}{g}}$, find the percentage change in t if l is increased by 1 °/$_\circ$.

Virtual Work

If a body or system of bodies is in equilibrium the work done by the external forces in any small displacement consistent with the geometrical conditions, i.e. any virtual displacement, is zero.

Such displacements are generally obtained by the Calculus, and by writing down the work done in such cases and equating it to zero we can find the position of equilibrium or obtain relations between the forces in that position.

Example 5.

Five light rods are freely jointed together so as to form a square $ABCD$ and one diagonal BD. It is suspended from A and a weight W hangs from C; show that the thrust in $BD=W$.

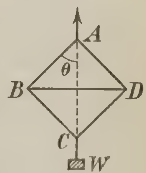

Fig. 154.

Let $\angle BAC=\theta$; then if a is the side of the square, $AC=2a\cos\theta$ and $BD=2a\sin\theta$.

By the principle of virtual work if T is the thrust in BD

$$Wd(AC)+Td(BD)=0;$$

i.e. $-Wa\sin\theta+Ta\cos\theta=0$,

$$\therefore\ T=W\tan\theta=W \qquad \text{since } \theta=\frac{\pi}{4}.$$

Example 6.

A string ACB passes over two pulleys A and B in a horizontal line 4 feet apart, it has weights of 5 lbs. at each end and 6 lbs. at C between the pulleys. Find the position of equilibrium.

If $\angle DCB=\theta$ we have $DC=2\cot\theta$ and $BC=2\operatorname{cosec}\theta$.

Imagine the 6 lbs. to descend a small distance, then the two 5 lbs. ascend and the work done by them is negative.

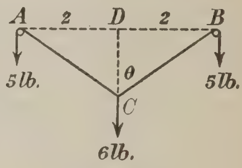

Fig. 155.

$$\therefore\ 6d(2\cot\theta)-10d(2\operatorname{cosec}\theta)=0,$$

$$\therefore\ -12\operatorname{cosec}^2\theta+20\operatorname{cosec}\theta\cot\theta=0,$$

$$\therefore\ \cos\theta=\tfrac{3}{5} \text{ for equilibrium.}$$

EXAMPLES XIV b

1. Four equal uniform rods each of weight W are smoothly hinged to form a rhombus $ABCD$. B and D are kept apart by a rod of weight w when the whole is suspended from A. If the angle DAB is $2a$, find the thrust in BD.

2. A ladder stands on a smooth floor with its legs both inclined at an angle θ to the vertical and their mid-points joined by a horizontal cord. If a weight W is placed on the top, find the additional tension in the cord.

3. A tripod made of three equal uniform rods freely jointed together stands on a smooth floor with the lower ends of the rods connected by strings equal in length to the rods. Find the tension in the strings if each rod weighs 2 lbs. and find also what the tension will be if 4 lbs. is placed on the vertex of the tripod.

4. A regular hexagon $ABCDEF$ of 6 equal rods, each of weight W, is freely jointed together. If it rests in a vertical plane with AB on a horizontal table, find the tension in a light string joining C to F.

Methods of Integration

Integration is largely a tentative process, but there are certain well-defined types of Integrals which admit of simple treatment which we now proceed to illustrate.

I. Integration at Sight

The following gives a list of Integrals which follow at once from the fact that Integrating is the reverse operation of Differentiating.

y	$\frac{dy}{dx}$	Integral
x^n	nx^{n-1}	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
$(ax+b)^n$	$an\,(ax+b)^{n-1}$	$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + c$
$\sin\,(ax+b)$	$a\cos\,(ax+b)$	$\int \cos\,(ax+b)\,dx = \frac{\sin\,(ax+b)}{a} + c$
$\cos\,(ax+b)$	$-a\sin\,(ax+b)$	$\int \sin\,(ax+b)\,dx = -\frac{\cos\,(ax+b)}{a} + c$
$\tan\,(ax+b)$	$a\sec^2(ax+b)$	$\int \sec^2(ax+b)\,dx = \frac{\tan\,(ax+b)}{a} + c$
$\cot\,(ax+b)$	$-a\,\mathrm{cosec}^2(ax+b)$	$\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{\cot\,(ax+b)}{a} + c$
e^{ax}	ae^{ax}	$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$
$\log_e x$	$\frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e x + c$
a^x	$a^x \log_e a$	$\int a^x\,dx = \frac{a^x}{\log_e a} + c$

Example 7.

Find $\int (2x+3)^3\,dx$.

As a tentative solution we write $(2x+3)^4$. Mental differentiation of this gives $8\,(2x+3)^3$.

$$\therefore\ \int (2x+3)^3\,dx = \tfrac{1}{8}\,(2x+3)^4 + c.$$

Example 8.

Integrate $\sin(3x+2)$.

We try $\cos(3x+2)$; mental differentiation gives $-3\sin(3x+2)$.

$$\therefore \int \sin(3x+2)\,dx = -\tfrac{1}{3}\cos(3x+2)+c.$$

Example 9.

Find $\int e^{-3x}\,dx$.

We try e^{-3x} which when differentiated gives $-3e^{-3x}$.

$$\therefore \int e^{-3x}\,dx = -\tfrac{1}{3}\int e^{-3x}+c.$$

Example 10.

Find $\int \frac{dx}{2-3x}$.

We try $\log_e(2-3x)$ which when differentiated gives $\frac{1}{2-3x}\times(-3)$.

$$\therefore \int \frac{dx}{2-3x} = \frac{\log_e(2-3x)}{-3}+c.$$

Note that the Denominator here must be *linear*, for $\frac{d}{dx}(\log_e x^2)$ is not $\frac{1}{x^2}$.

EXAMPLES XIV c

Find

1. $\int (x+4)^2\,dx$. **2.** $\int (2-x)^3\,dx$. **3.** $\int (2x+3)^{\frac{1}{2}}\,dx$.

4. $\int \frac{dx}{(2x+1)^3}$. **5.** $\int \sqrt{2x+5}\,dx$. **6.** $\int \sqrt{ax+b}\,dx$.

7. $\int \cos(2x-5)\,dx$. **8.** $\int \sin(4-x)\,dx$. **9.** $\int \sec^2(nx+a)\,dx$.

10. $\int \sin^2 x\,d(\sin x)$. **11.** $\int \sin^2 x\cos x\,dx$. **12.** $\int \tan x\,d(\tan x)$.

13. $\int \tan x\sec^2 x\,dx$. **14.** $\int \cot x\,\mathrm{cosec}^2 x\,dx$. **15.** $\int e^{-ax}\,dx$.

16. $\int e^{(ax+b)}\,dx$. **17.** $\int \frac{dx}{2x-3}$. **18.** $\frac{dx}{ax+b}$.

19. $\int \frac{dx}{2-x}$. **20.** $\int x^{0\cdot4}\,dx$. **21.** $\int_2^1 \frac{dv}{v^{1\cdot4}}$. **22.** $\int x^{-0\cdot4}\,dx$.

23. $\int (x+1)\sqrt{(x^2+2x+3)}\,dx$. **24.** $\int \frac{\sin x}{(1+2\cos x)^2}\,dx$. **25.** $\int xe^{-x^2}\,dx$.

II. Change of variable by substitution

If $z = \int f(x)\,dx$, then $\frac{dz}{dx} = f(x)$.

Now substitute $x = \phi(u)$, i.e. $\frac{dx}{du} = \phi'(u)$, and express the integral in terms of u as the variable

$$\frac{dz}{du} = \frac{dz}{dx} \times \frac{dx}{du} = \frac{dz}{dx} \times \phi'(u)$$

$$= f(x) \times \phi'(u);$$

$$\therefore\; z = \int f(x)\,\phi'(u)\,du.$$

That is to say, we may write for dx the differential $\phi'(u)\,du$ and then integrate with respect to u.

We shall consider two special cases:

(a) *Reduction to the form* $\int x^n\,dx$.

Example 11.

Find $\int x\sqrt{x^2+1}\,dx$.

Let $z = \int x\sqrt{x^2+1}\,dx$, then $\frac{dz}{dx} = x\sqrt{x^2+1}$. Substitute $u = x^2+1$, then $\frac{du}{dx} = 2x$.

Changing from $\frac{dz}{dx}$ to $\frac{dz}{du}$ we have

$$\frac{dz}{du} = \frac{dz}{dx} \times \frac{dx}{du}$$

$$= x\sqrt{x^2+1} \times \frac{1}{2x} = \frac{\sqrt{x^2+1}}{2} = \frac{u^{\frac{1}{2}}}{2};$$

$$\therefore\; z = \tfrac{1}{2}\int u^{\frac{1}{2}}\,du + c = \tfrac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{(x^2+1)^{\frac{3}{2}}}{3} + c.$$

The correctness of this result should be tested by finding $\frac{dz}{dx}$ from it.

We may however proceed more shortly as follows:

By direct application of the result proved above that

$$\int f(x)\,dx = \int f(x)\,\phi'(u)\,du,$$

we have, by putting $u = x^2+1$,

$$\frac{du}{dx} = 2x \quad \text{or} \quad du = 2x\,dx;$$

$$\therefore \int x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\,(x\,dx)$$

$$= \int u^{\frac{1}{2}}\frac{du}{2} = \tfrac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{(x^2+1)^{\frac{3}{2}}}{3} + c.$$

Note. When we change the variable in a definite integral we must of course also change the limits, and since the result of a definite integral is a function of the limits and not a function of the variable, there is no need to substitute back to the original variable. This will be clear if we integrate the last example from $x=0$ to $x=1$. Then

$$\int_0^1 x\sqrt{x^2+1}\,dx = \tfrac{1}{2}\int_1^2 u^{\frac{1}{2}}\,du.$$

For since $u = x^2+1$, when $x=0$, u will equal 1,

and when $x=1$, u will equal 2.

$\therefore$ the limits for u are from 1 to 2.

$$\therefore \text{integral} = \tfrac{1}{3}\Big[u^{\frac{3}{2}}\Big]_1^2 = \frac{(\sqrt{8}-1)}{3}.$$

(*b*) *Reduction to the form* $\int \frac{dx}{x}$.

Example 12.

Find $\int \frac{x}{x^2+1}\,dx$.

We notice that the numerator x equals the derivative of the denominator multiplied by a constant; this suggests the form $\int \frac{dx}{x}$.

Put $u = x^2+1$, then $du = 2x\,dx$;

$$\therefore \int \frac{x}{x^2+1}\,dx = \tfrac{1}{2}\int \frac{du}{u} = \tfrac{1}{2}\log_e u + c$$

$$= \tfrac{1}{2}\log_e(x^2+1) + c.$$

Example 13.

Find $\int \tan x\, dx$.

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\log_e \cos x + c$$
$$= \log_e \sec x + c.$$

EXAMPLES XIV d

Evaluate:

1. $\int x\sqrt{3-x^2}\, dx.$
2. $\int \sqrt{a^2-x^2}\, x\, dx.$
3. $\int \frac{dx}{\sqrt{1-x}}.$
4. $\int \frac{x\, dx}{\sqrt{1-x^2}}.$
5. $\int \frac{x^2\, dx}{\sqrt{a^2+x^3}}.$
6. $\int 2\pi y\sqrt{y^2+a^2}\, dy.$
7. $\int_0^1 \frac{dx}{\sqrt{2-x}}.$
8. $\int_2^3 \frac{x\, dx}{x^2-1}.$
9. $\int \frac{t^2\, dt}{3+t^3}.$
10. $\int \frac{3ax\, dx}{b^2+a^2x^2}.$
11. $\int \frac{1}{3-x}\, dx.$
12. $\int \frac{1}{a-bx}\, dx.$
13. $\int \cot x\, dx.$
14. $\int \frac{(3x^2-2x)\, dx}{x^3-x^2+1}.$
15. $\int \frac{1}{x}\log x\, dx.$
16. $\int x^3\sqrt{(1+x^4)}\, dx.$
17. $\int \sin^3 x \cos x\, dx.$
18. $\int_0^{\frac{\pi}{3}} \frac{\sin\theta\, d\theta}{3+4\cos\theta}.$
19. $\int \tan^5\theta \sec^2\theta\, d\theta.$

III. Powers of $\sin x$ or $\cos x$

In order to integrate powers of $\sin x$ or $\cos x$ or products of $\sin x$ and $\cos x$ it is usually necessary to express them as functions of the first degree, using multiple angles.

Example 14.

Find (i) $\int \sin^2 x\, dx$; (ii) $\int \cos^3 x\, dx$.

(i) We have $\cos 2x = 1 - 2\sin^2 x$ or $\sin^2 x = \frac{1-\cos 2x}{2}$,

$$\therefore \int \sin^2 x\, dx = \tfrac{1}{2}\int (1-\cos 2x)\, dx = \tfrac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c.$$

(ii) We have $\cos 3x = 4\cos^3 x - 3\cos x$ or $\cos^3 x = \tfrac{1}{4}(\cos 3x + 3\cos x)$,

$$\therefore \int \cos^3 x\, dx = \tfrac{1}{4}\int (\cos 3x + 3\cos x)\, dx = \tfrac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c.$$

IV. Trigonometrical substitutions

It is often possible to simplify expressions by using a trigonometrical substitution; e.g. if the expression involves $\sqrt{(a^2-x^2)}$, and if we put $x=a\sin\theta$, we have

$$\sqrt{(a^2-x^2)}=\sqrt{(a^2-a^2\sin^2\theta)}=\sqrt{(a^2\cos^2\theta)}=a\cos\theta;$$

or if the expression involves (a^2+x^2), it may be useful to put $x=a\tan\theta$, then $\quad a^2+x^2=a^2+a^2\tan^2\theta=a^2\sec^2\theta;$

or more generally in $\sqrt{[a^2-(x+b)^2]}$, put $x+b=a\sin\theta$.

The cases $\sqrt{(x^2-a^2)}$ and $\sqrt{x^2+a^2}$ will be considered later (see p. 272).

Example 15.

Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{(9-x^2)}}$.

Put $\qquad x=3\sin\theta,\ \therefore dx=3\cos\theta\,d\theta,$

and $\qquad \sqrt{(9-x^2)}=\sqrt{(9-9\sin^2\theta)}=\sqrt{(9\cos^2\theta)}=3\cos\theta.$

Also when $x=0$, $\theta=0$ and when $x=3$, $\sin\theta=1$ and $\theta=\frac{\pi}{2}$.

$$\therefore \text{ expression}=\int_0^{\frac{\pi}{2}}\frac{3\cos\theta\,d\theta}{3\cos\theta}=\int_0^{\frac{\pi}{2}}d\theta=\Big[\theta\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2}.$$

Note. This integral could be evaluated more shortly by using the fact

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]=\frac{1}{\sqrt{(a^2-x^2)}}.$$

Example 16.

Evaluate $\qquad \displaystyle\int\sqrt{a^2-x^2}\,dx.$

Here x is less than a, so we try $x=a\sin\theta$ then $dx=a\cos\theta\,d\theta$.

$$\begin{aligned}\therefore \int\sqrt{a^2-x^2}\,dx&=\int a\cos\theta\,.\,a\cos\theta\,d\theta=a^2\int\cos^2\theta\,d\theta\\ &=\frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta=\frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c\\ &=\frac{a^2}{2}[\theta+\sin\theta\cos\theta]+c\\ &=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}+c\\ &=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\tfrac{1}{2}x\sqrt{a^2-x^2}+c.\end{aligned}$$

If we put $y=\sqrt{a^2-x^2}$ we have $y^2+x^2=a^2$ which is the equation of a circle with origin at the centre, and if the Integral is taken from $x=0$ to $x=x\,(ON)$ we see that it is made up of the sector $BOP+\Delta\,ONP$ whose area is $\frac{1}{2}a^2\theta+\frac{1}{2}xy$ where $\angle\,PON=\theta$

$$=\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}+\tfrac{1}{2}x\sqrt{a^2-x^2}.$$

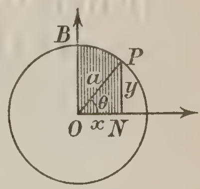

Fig. 156.

If we require $\int_0^a \sqrt{a^2-x^2}\,dx$ we have as limits for θ, $\theta=0$ when $x=0$, and $\theta=\frac{\pi}{2}$ when $x=a$.

$$\therefore\ \int_0^a \sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{a^2}{2}\left(\frac{\pi}{2}\right)=\frac{a^2\pi}{4}.$$

This is the area of a quadrant, $\therefore$ area of whole circle $=\pi a^2$.

Example 17.

Evaluate
$$\int\frac{dx}{\sqrt{(4+5x-3x^2)}}.$$

We reduce $4+5x-3x^2$ to the form $a^2-(b+x)^2$.

$$4+5x-3x^2=4-3\left(x^2-\frac{5x}{3}\right)=4-3\left[x^2-\frac{5x}{3}+\left(\frac{5}{6}\right)^2\right]+\frac{25}{12}$$

$$=\frac{73}{12}-3\left(x-\frac{5}{6}\right)^2=3\left\{\frac{73}{36}-\left(x-\frac{5}{6}\right)^2\right\}.$$

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\frac{73}{36}-\left(x-\frac{5}{6}\right)^2\right\}}},$$

put
$$x-\frac{5}{6}=\frac{\sqrt{73}}{6}\sin\theta,\qquad \therefore\ dx=\frac{\sqrt{73}}{6}\cos\theta\,d\theta,$$

and the denominator $=\sqrt{\left(\frac{73}{36}-\frac{73}{36}\sin^2\theta\right)}=\sqrt{\left(\frac{73}{36}\cos^2\theta\right)}=\frac{\sqrt{73}}{6}\cos\theta$,

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\int\frac{\frac{\sqrt{73}}{6}\cos\theta\,d\theta}{\frac{\sqrt{73}}{6}\cos\theta}=\frac{1}{\sqrt{3}}\int d\theta=\frac{\theta}{\sqrt{3}}+c.$$

But $\sin\theta=\dfrac{6x-5}{\sqrt{73}}$ or $\theta=\sin^{-1}\left(\dfrac{6x-5}{\sqrt{73}}\right)$,

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{73}}\right)+c.$$

Example 18.

Evaluate $$\int \frac{dx}{4+x^2}.$$

Put $x=2\tan\theta$, then $dx=2\sec^2\theta\,d\theta$ and $4+x^2=4+4\tan^2\theta=4\sec^2\theta$.

$$\therefore \text{ expression} = \int \frac{2\sec^2\theta\,d\theta}{4\sec^2\theta} = \tfrac{1}{2}\int d\theta = \frac{\theta}{2} = \tfrac{1}{2}\tan^{-1}\left(\frac{x}{2}\right).$$

Note. This result could have been obtained more shortly by using

$$\frac{d}{dx}\left[\tan^{-1}\frac{x}{a}\right] = \frac{a}{a^2+x^2}.$$

Example 19.

Find x when $\dfrac{d^2x}{dt^2}+\mu x=0$, given that $\dfrac{dx}{dt}=0$ when $x=a$ and that $x=a$ when $t=0$.

This example is of importance in Harmonic Motion.

Multiply both sides by $2\dfrac{dx}{dt}$, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\mu x\frac{dx}{dt}=0,$$

i.e.
$$\frac{d}{dt}\left(\frac{dx}{dt}\right)^2+\mu\frac{d}{dt}(x^2)=0;$$

$$\therefore \left(\frac{dx}{dt}\right)^2+\mu x^2=c.$$

Since $\dfrac{dx}{dt}=0$ when $x=a$, we find $c=\mu a^2$.

$$\therefore \frac{dx}{dt}=\pm\sqrt{\mu}\sqrt{a^2-x^2}.$$

Taking the case when the velocity is negative, we have

$$dt=-\frac{dx}{\sqrt{\mu}\sqrt{a^2-x^2}}.$$

By substituting $x=a\cos\theta$ we get

$$t=\frac{1}{\sqrt{\mu}}\theta+b \text{ and } b=0 \text{ for } x=a \text{ and } \theta=0 \text{ when } t=0.$$

$$\therefore \sqrt{\mu}\,t=\cos^{-1}\frac{x}{a} \text{ or } x=a\cos(\sqrt{\mu}\,.\,t).$$

EXAMPLES XIV e

Evaluate the following integrals:

1. $\int \cos^2 x\,dx.$ 2. $\int \cos^2 3x\,dx.$ 3. $\int \sin ax \cos bx\,dx.$

4. $\int \tan^2 x\,dx.$ 5. $\int \sin(1-x)\,dx.$ 6. $\int_0^{0\cdot 8} \sin^2 x \cos x\,dx.$

7. $\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt.$ 8. $\int \sin^3 x\,dx.$ 9. $\int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x\,dx.$

10. $\int \frac{dx}{\sqrt{(16-x^2)}}.$ 11. $\int_0^2 \sqrt{(4-x^2)}\,dx.$ 12. $\int \frac{dx}{x^2+3}.$

13. $\int_0^a \frac{dx}{a^2+x^2}.$ 14. $\int \frac{dx}{x^2+6x+10}.$ 15. $\int \frac{dx}{\sqrt{(3+2x-x^2)}}.$

16. $\int \sqrt{(7-3x^2)}\,dx.$ 17. $\int_3^4 \sqrt{(25-x^2)}\,dx.$ 18. $\int \frac{dx}{\sqrt{(ax-x^2)}}.$

19. $\int \frac{dx}{\sqrt{(6x-x^2-8)}}.$ 20. $\int \cot^2(ax+b)\,dx.$ 21. $\int_0^1 (1-x^2)^{\frac{3}{2}}\,dx.$

22. If
$$u=\int_0^T \sin\frac{2\pi rx}{T}\cos\frac{2\pi sx}{T}\,dx,$$
$$v=\int_0^T \sin\frac{2\pi rx}{T}\sin\frac{2\pi sx}{T}\,dx,$$
$$w=\int_0^T \cos\frac{2\pi rx}{T}\cos\frac{2\pi sx}{T}\,dx,$$
where r, s are integers, prove that

(i) $u=0$, (ii) $v=w=0$ if $r \neq s$, (iii) $v=w=\frac{T}{2}$ if $r=s$.

23. Draw the curve $y=\sqrt{4-x^2}\,dx$ and find the values of $\int_0^1 \sqrt{4-x^2}\,dx$ and $\int_1^2 \sqrt{4-x^2}\,dx$.

Show without using the Calculus that the results should be $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$ and $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$.

24. Show that the curve $y=\sqrt{x(1-x)}$ is a circle whose centre is at $(\frac{1}{2}, 0)$; hence evaluate $\int_0^{\frac{1}{2}} \sqrt{x(1-x)}\, dx$. Find this area also by integration.

25. Plot the graph of $\frac{1}{1+x^2}$ and find $\int_0^1 \frac{1}{1+x^2}\, dx$. Evaluate this area also by Simpson's Rule and hence find π.

26. A conductor rotates uniformly about a fixed axis while an electric current x flows along it such that $x=k\sin\theta$, where θ is the angle through which the conductor has turned. Find the maximum value of the current and the average value during the time it is positive. If a^2 is the average value of $k^2\sin^2\theta$, a is called the virtual current. Show that it equals about $0{\cdot}71\,k$.

V. Rational Functions of x

(*a*) When the degree of the numerator is equal to or greater than the degree of the denominator.

Divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

Example 20.
Evaluate
$$\int \frac{x^2-x}{x+1}\, dx.$$

Now
$$\frac{x^2-x}{x+1} = x-2+\frac{2}{x+1},$$

$$\therefore \text{ integral} = \int \left[x-2+\frac{2}{x+1}\right] dx$$
$$= \tfrac{1}{2}x^2 - 2x + 2\log(x+1) + c.$$

(*b*) When the denominator is quadratic and has no real factors.

If the numerator contains x, this must first be removed by the substitution method.

Example 21.
Evaluate
$$\int \frac{5x-2}{x^2+6x+13}\, dx.$$

Now
$$\frac{d}{dx}(x^2+6x+13) = 2x+6;$$

we therefore express $5x-2$ in the form

$$5x-2 \equiv \tfrac{5}{2}(2x+6) - 15 - 2 \equiv \tfrac{5}{2}(2x+6) - 17;$$

$$\therefore \text{ expression} = \tfrac{5}{2}\int\frac{2x+6}{x^2+6x+13}\,dx - 17\int\frac{dx}{x^2+6x+13}$$

$$= \tfrac{5}{2}\int\frac{d\,(x^2+6x+13)}{x^2+6x+13} - 17\int\frac{dx}{(x+3)^2+4}.$$

The first integral $= \frac{5}{2}\log(x^2+6x+13)$.

In the second integral, put $x+3=2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$.

Then $\displaystyle\int\frac{dx}{(x+3)^2+4} = \int\frac{2\sec^2\theta\,d\theta}{4\tan^2\theta+4} = \tfrac{1}{2}\int d\theta = \frac{\theta}{2} = \tfrac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$;

$$\therefore \text{ expression} = \tfrac{5}{2}\log(x^2+6x+13) - \tfrac{17}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c.$$

Note. If $ax^2+2bx+c$ has no real factors, it can be expressed in the form

$$a\left(x^2+\frac{2b}{a}x+\frac{b^2}{a^2}\right)+c-\frac{b^2}{a},$$

or
$$a\left\{\left(x+\frac{b}{a}\right)^2+\frac{ac-b^2}{a^2}\right\},$$

or
$$a\{(x+p)^2+q^2\};$$

and $\displaystyle\int\frac{dx}{(x+p)^2+q^2}$ is evaluated by putting $x+p=q\tan\theta$, its value is

$$\frac{1}{q}\tan^{-1}\left(\frac{x+p}{q}\right).$$

Thus

$$\int\frac{dx}{x^2+x+1} = \int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int\frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

(*c*) When the denominator is a product of distinct linear factors. Split up the expression into partial fractions.

Example 22.

Evaluate
$$\int\frac{2x-7}{(x-2)(x-3)}\,dx.$$

Let $\dfrac{2x-7}{(x-2)(x-3)} \equiv \dfrac{A}{x-2}+\dfrac{B}{x-3}$, where A, B are constants;

$$\therefore\ 2x-7 \equiv A(x-3)+B(x-2);$$

this is true for *all* values of x.

$\therefore$ when $x=2$, $\quad 4-7=-A$ or $A=3$;

when $x=3$, $\quad 6-7=B \quad$ or $B=-1$.

$$\therefore \text{ expression} = \int\left(\frac{3}{x-2}-\frac{1}{x-3}\right)dx$$
$$=3\log(x-2)-\log(x-3)+c$$
$$=\log\frac{(x-2)^3}{x-3}+c \text{ or } \log\left\{\frac{a(x-2)^3}{x-3}\right\}.$$

Note.

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx=\frac{1}{2a}\Big[-\log(a-x)+\log(a+x)\Big]+c$$
$$=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c.$$

(*d*) When the denominator contains repeated linear factors. As before, split up the expression into partial fractions.

Example 23.

Evaluate $$\int\frac{x-5}{(x+2)(x+3)^2}dx.$$

$\dfrac{x-5}{(x+2)(x+3)^2}$ can be expressed in the form $\dfrac{A}{x+2}+\dfrac{qx+r}{(x+3)^2}$.

Since the denominator $(x+3)^2$ is of the second degree, the numerator could contain x, and $\therefore$ the term qx cannot be omitted.

Now

$$\frac{qx+r}{(x+3)^2}=\frac{q(x+3)+r-3q}{(x+3)^2}=\frac{q(x+3)}{(x+3)^2}+\frac{r-3q}{(x+3)^2}=\frac{q}{x+3}+\frac{r-3q}{(x+3)^2}.$$

$\therefore \dfrac{qx+r}{(x+3)^2}$ can be written in the form $\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$.

$\therefore \dfrac{x-5}{(x+2)(x+3)^2}$ can be put $\equiv\dfrac{A}{x+2}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$ where A, B, C are constants.

$\therefore x-5\equiv A(x+3)^2+B(x+2)(x+3)+C(x+2)$ for *all* values of x.

$\therefore$ when $x=-3$, $\quad -3-5=-C$ or $C=8$,

when $x=-2$, $\quad -2-5=A \quad$ or $A=-7$,

equate coefficients of x^2,

$$\therefore 0=A+B \text{ or } B=-A=7;$$

$$\therefore \text{ expression} = \int\left[\frac{-7}{x+2}+\frac{7}{x+3}+\frac{8}{(x+3)^2}\right]dx$$

$$= -7\log(x+2)+7\log(x+3)-\frac{8}{x+3}+c$$

$$= \log\left(\frac{x+3}{x+2}\right)^7 - \frac{8}{x+3}+c.$$

(*e*) When the denominator contains both linear and irreducible quadratic factors.

Split up the expression into partial fractions and proceed as before.

Example 24.

Evaluate $$\int\frac{3x^2+14}{(x+2)(x^2-2x+5)}\,dx.$$

Let $$\frac{3x^2+14}{(x+2)(x^2-2x+5)} \equiv \frac{A}{x+2}+\frac{Bx+C}{x^2-2x+5},$$

$$\therefore\ 3x^2+14 \equiv A(x^2-2x+5)+(Bx+C)(x+2),$$

put $x=-2$, $\quad\therefore\ 12+14=13A$ or $A=2$,

put $x=0$, $\quad\therefore\ 14=5A+2C$ or $2C=14-10$, $C=2$,

equate coefficients of x^2, $\therefore\ 3=A+B$ or $B=1$;

$$\therefore \text{ expression} = \int\left(\frac{2}{x+2}+\frac{x+2}{x^2-2x+5}\right)dx.$$

Now $\frac{d}{dx}(x^2-2x+5)=2x-2$, $\therefore$ write $x+2=\frac{1}{2}(2x-2)+3$;

$$\therefore \text{ expression} = 2\int\frac{dx}{x+2}+\frac{1}{2}\int\frac{2x-2}{x^2-2x+5}\,dx+3\int\frac{dx}{(x-1)^2+4}$$

$$= 2\log(x+2)+\frac{1}{2}\log(x^2-2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right).$$

EXAMPLES XIV f

Evaluate:

1. $\int\frac{x}{1+x}\,dx.$
2. $\int\frac{x+2}{x+3}\,dx.$
3. $\int\frac{x^2+3}{x+2}\,dx.$
4. $\int\frac{dx}{x^2-1}.$
5. $\int\frac{dx}{4x^2-9}.$
6. $\int\frac{x\,dx}{x^2-4}.$
7. $\int\frac{dx}{4x^2+1}.$
8. $\int\frac{3x+2}{4x^2+1}\,dx.$
9. $\int\frac{1-x}{x^2+16}\,dx.$
10. $\int\frac{dx}{x^2-2x+10}.$
11. $\int\frac{5x}{x^2-2x+10}\,dx.$
12. $\int\frac{x+1}{(x+a)^2+b^2}\,dx.$

13. $\displaystyle\int\frac{dx}{(x+1)(x+2)}$. **14.** $\displaystyle\int\frac{x+1}{(x-2)(x-3)}\,dx$.

15. $\displaystyle\int\frac{1+5x}{(1-x)(2+x)}\,dx$. **16.** $\displaystyle\int\frac{1+3x}{10-3x-x^2}\,dx$. **17.** $\displaystyle\int\frac{dx}{x^2(x-1)}$.

18. $\displaystyle\int\frac{5-2x}{(x-1)^2(x+2)}\,dx$. **19.** $\displaystyle\int\frac{2x^3+3x^2+1}{x+2}\,dx$.

20. $\displaystyle\int\frac{30x+51}{(x-2)(x^2+8x+17)}\,dx$. **21.** $\displaystyle\int\frac{dx}{(x-2)(x^2+3x+1)}$.

22. $\displaystyle\int\frac{(x-4)\,dx}{(x-1)(x-2)(x-3)}$. **23.** $\displaystyle\int\frac{x^2\,dx}{(x+1)^3}$.

24. $\displaystyle\int\frac{dx}{x^3+x^2+x}$.

25. Find the area between the x-axis, the curve $y=\dfrac{x}{x+1}$ and the ordinates $x=0$ and $x=1$.

26. Evaluate $\displaystyle\int_0^1\frac{dx}{x^2+x+1}$.

27. Evaluate $\displaystyle\int_3^4\frac{x^2\,dx}{x-2}$.

28. Evaluate $\displaystyle\int_1^a\frac{dx}{x(x+1)^2}$, and find its limit as $a\to\infty$.

29. Find the equation of the curve whose gradient at the point (x, y) is $\dfrac{x}{x+2}$, given that the point $(-1, 0)$ lies on the curve.

30. If a motor-car is driven by a constant force and if the air resistance varies as the square of the speed v, then $\dfrac{dv}{dt}=k(V^2-v^2)$, where k, V are constants; find the time taken starting from rest to acquire a speed $\frac{1}{2}V$.

VI. Integration by Parts

Since $d(uv)=u\,dv+v\,du$, we have by integrating both sides $uv=\int u\,dv+\int v\,du$.

$$\therefore\ \int u\,dv=uv-\int v\,du.$$

The success of this method depends upon the possibility of finding $\int v\,du$ instead of $\int u\,dv$.

Example 25.

$$\int x\cos x\,dx=\int x\,d(\sin x) \qquad \text{i.e. } x=u \text{ and } \sin x=v$$

$$=x\sin x-\int \sin x\,dx$$

$$=x\sin x+\cos x+c.$$

This might have been written $\int \cos x\,d\left(\frac{x^2}{2}\right)$ so that $u=\cos x$ and $v=\frac{x^2}{2}$.

Then
$$\int \cos x\,d\left(\frac{x^2}{2}\right)=\cos x\,\frac{x^2}{2}-\int \frac{x^2}{2}\,d(\cos x)$$

$$=\frac{x^2\cos x}{2}+\tfrac{1}{2}\int x^2\sin x\,dx.$$

But it is no easier to find $\int x^2\sin x\,dx$ than to find the original integral, so that it is necessary to make a trial to find which arrangement will be successful.

Example 26.
$$\int \log x\,dx=(\log x)(x)-\int x\frac{1}{x}\,dx$$

$$=x\log x-x+c.$$

Example 27.
$$z=\int e^{ax}\sin bx=\frac{1}{a}\int \sin bx\,d(e^{ax})$$

$$=\frac{1}{a}\sin bx\,e^{ax}-\frac{b}{a}\int e^{ax}\cos bx\,dx.$$

But
$$\int e^{ax}\cos bx\,dx=\frac{1}{a}\int \cos bx\,d(e^{ax})$$

$$=\frac{1}{a}\cos bx\,e^{ax}+\frac{b}{a}\int e^{ax}\sin bx\,dx.$$

$$\therefore\ z=\frac{1}{a}\sin bx\,e^{ax}-\frac{b}{a}\left[\frac{1}{a}\cos bx\,e^{ax}+\frac{b}{a}\int e^{ax}\sin bx\,dx\right],$$

$$\therefore\ z=\frac{1}{a}\sin bx\,e^{ax}-\frac{b}{a^2}\cos bx\,e^{ax}-\frac{b^2}{a^2}z,$$

$$\therefore\ z\left(1+\frac{b^2}{a^2}\right)=\frac{e^{ax}[a\sin bx-b\cos bx]}{a^2},$$

$$\therefore\ z=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}.$$

EXAMPLES XIV g

Integrate the expressions in Examples 1—18:

1. $x \log x$. **2.** $x e^x$. **3.** $x^2 \log x$.

4. $x \sin x$. **5.** $x(1+x)^{10}$. **6.** $x \cos 3x$.

7. $x \sec^2 x$. **8.** $x \tan^{-1} x$. **9.** $x \sin x \cos x$.

10. $\sin^{-1} x\, dx$. **11.** $x^2 \cos x$. **12.** $\dfrac{\log x}{x^3}$.

13. $e^x \sin x$. **14.** $e^x \cos x$. **15.** $\sin^5 x = -\sin^4 x\, d(\cos x)$.

16. $x^n \log x$. **17.** $e^{-2x} \sin 3x$. **18.** $x^2(1-x)^{20}$.

19. Show that

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx.$$

This is called a formula of reduction.

20. Prove $\displaystyle\int x^m \sin x\, dx = -x^m \cos x + m\int x^{m-1} \cos x\, dx.$

Hence find $\displaystyle\int x^5 \sin x\, dx.$

21. Use Example 19 to prove that

$$n\int_0^{\frac{\pi}{2}} \sin^n x\, dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx,$$

and hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x\, dx.$

22. Obtain a formula of reduction for $\int \cos^n x\, dx$ similar to the result in Example 19; and prove that

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx.$$

23. Obtain a formula of reduction for $\int x^n e^{-x}\, dx$ and hence evaluate

$$\int_0^{\infty} x^{10} e^{-x}\, dx.$$

24. Prove that

$$\int \sin^m \theta \cos^n \theta\, d\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n \theta\, d\theta.$$

Hence, using also Example 22, evaluate

$$\int_0^{\frac{\pi}{2}} \sin^6 \theta \cos^4 \theta\, d\theta \text{ and } \int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^4 \theta\, d\theta.$$

VII. Additional Integrals

(i) Any rational function of $\sin\theta$ and $\cos\theta$ can be expressed as a rational function of $t \equiv \tan\frac{\theta}{2}$,

for
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}} = \frac{2t}{1+t^2},$$

and
$$\cos\theta = \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2},$$

and $dt = d\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1}{2}(1+t^2)\,d\theta, \quad \therefore\ d\theta = \frac{2dt}{1+t^2}.$

Consequently any rational function of $\sin\theta$ and $\cos\theta$ can be reduced to the type dealt with in V.

The following special cases are of importance:

(a)
$$\int\frac{d\theta}{\sin\theta} = \int\frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \int\frac{\sec^2\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}}$$
$$= \int\frac{dt}{t} \qquad \text{where } t \equiv \tan\frac{\theta}{2}$$
$$= \log t = \log\left(\tan\frac{\theta}{2}\right) + c.$$

(b)
$$\int\frac{d\theta}{\cos\theta} = \int\frac{d\theta}{\sin\left(\frac{\pi}{2}+\theta\right)} = \int\frac{d\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)}$$
$$= \log\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\} + c \qquad \text{from } (a).$$

(c) $\displaystyle\int\frac{d\theta}{a+b\cos\theta}$, put $t = \tan\frac{\theta}{2}$ and proceed as above.

$$\therefore\ \text{expression} = \int\frac{\frac{2dt}{1+t^2}}{a+\frac{b(1-t^2)}{1+t^2}} = 2\int\frac{dt}{(a+b)+(a-b)t^2}.$$

Suppose $a > b$, then it

$$= \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b} + t^2}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left\{t\sqrt{\left(\frac{a-b}{a+b}\right)}\right\} + c$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\tan\frac{\theta}{2}\sqrt{\left(\frac{a-b}{a+b}\right)}\right\} + c.$$

If $a < b$, the integral involves a logarithm and is effected by partial fractions.

(ii) Functions involving $\sqrt{(x^2+a^2)}$ or $\sqrt{(x^2-a^2)}$ are best treated by hyperbolic substitutions, see p. 272; but the following methods are sometimes used.

(*a*) $$\int \frac{dx}{\sqrt{(x^2+a^2)}}.$$

Put $z - x = \sqrt{(x^2+a^2)}$;

$$\therefore\ (z-x)^2 = x^2 + a^2 \text{ or } z^2 - 2zx + x^2 = x^2 + a^2,$$

$$\therefore\ z^2 - 2zx = a^2,$$

$$\therefore\ 2z\,dz - 2(z\,dx + x\,dz) = 0,$$

$$\therefore\ dz\,(z-x) = z\,dx,$$

$$\therefore\ \frac{dz}{z} = \frac{dx}{z-x},$$

$$\therefore\ \int\frac{dx}{\sqrt{(x^2+a^2)}} = \int\frac{dx}{z-x} = \int\frac{dz}{z} = \log z + c$$

$$= \log\left[x + \sqrt{(x^2+a^2)}\right] + c.$$

(*b*) $$\int\sqrt{(x^2+a^2)}\,dx.$$

By parts, $\displaystyle\int\sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} - \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}}.$

But

$$\int\sqrt{(x^2+a^2)}\,dx = \int\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\,dx = \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}} + a^2\int\frac{dx}{\sqrt{(x^2+a^2)}};$$

$$\therefore \text{ adding } \; 2\int \sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} + a^2\int \frac{dx}{\sqrt{(x^2+a^2)}},$$

$$\therefore \int \sqrt{(x^2+a^2)}\,dx = \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{a^2}{2}\log\left[x+\sqrt{(x^2+a^2)}\right] + c.$$

(iii) There are certain integrals which for special limits can be easily calculated; we shall discuss one important definite integral of this kind.

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta.$$

For brevity, write $\sin\theta \equiv s$ and $\cos\theta \equiv c$.

$$\begin{aligned} \text{Now } \; \frac{d}{d\theta}(s^{m+1}c^{n-1}) &= (m+1)\,s^m c^n - (n-1)\,s^{m+2}c^{n-2} \\ &= (m+1)\,s^m c^n - (n-1)\,s^m c^{n-2}(1-c^2) \\ &= (m+n)\,s^m c^n - (n-1)\,s^m c^{n-2}. \end{aligned}$$

Integrate this from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Since $s = 0$ if $\theta = 0$ and $c = 0$ if $\theta = \frac{\pi}{2}$, $\left[s^{m+1}c^{n-1}\right]_0^{\frac{\pi}{2}} = 0$,

$$\therefore \int_0^{\frac{\pi}{2}} s^m c^n \, d\theta = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} s^m c^{n-2}\, d\theta \quad \text{.....................(1)}$$

$$= \text{by symmetry } \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} s^{m-2}c^n\, d\theta \quad \text{...(2).}$$

Put $m = 0$ in (1), then

$$\int_0^{\frac{\pi}{2}} c^n\, d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} c^{n-2}\, d\theta \quad \text{..............(3).}$$

Put $n = 0$ in (2), then

$$\int_0^{\frac{\pi}{2}} s^m\, d\theta = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} s^{m-2} d\theta \quad \text{..............(4).}$$

The results in (1), (2), (3), (4) enable us to evaluate the integral in any case: we shall illustrate the various possibilities by examples.

Example 28.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta \text{ and } \int_0^{\frac{\pi}{2}} \cos^8\theta\,d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta = \frac{7}{8}\int_0^{\frac{\pi}{2}} \sin^6\theta\,d\theta = \frac{7}{8}\times\frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$

$$= \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}.$$

$\int_0^{\frac{\pi}{2}} \cos^8\theta\,d\theta$ has the same value.

Example 29.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta = \frac{6}{7}\int_0^{\frac{\pi}{2}} \sin^5\theta\,d\theta = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta$$

$$= \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}.$$

Note the difference in form of the answer according as whether the index is even or odd.

Example 30.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\,d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\,d\theta = \frac{3}{10}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^6\theta\,d\theta = \frac{3}{10}\times\frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^6\theta\,d\theta$$

$$= \frac{3}{10}\times\frac{1}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{3}{10}\times\frac{1}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}.$$

Example 31.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta\,d\theta.$$

If either index is *odd*, reduce that one.

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^5\theta \, d\theta = \frac{4}{9}\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^3\theta\,d\theta = \frac{4}{9}\times\frac{2}{7}\int_0^{\frac{\pi}{2}}\sin^4\theta\cos\theta\,d\theta$$

$$= \frac{4}{9}\times\frac{2}{7}\int_0^{\frac{\pi}{2}}\sin^4\theta\, d(\sin\theta) = \frac{4}{9}\times\frac{2}{7}\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4}{9}\times\frac{2}{7}\times\frac{1}{5}.$$

Example 32.

Evaluate $$\int_0^1 x^2(1-x^2)^{\frac{3}{2}}\,dx.$$

Put $x = \sin\theta$, then expression

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,d\theta$$

$$= \frac{1}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{32}.$$

EXAMPLES XIV h

Evaluate

1. $\displaystyle\int\frac{d\theta}{\sin\frac{\theta}{2}}$. **2.** $\displaystyle\int\frac{d\theta}{\cos\frac{\theta}{2}}$. **3.** $\displaystyle\int\frac{d\theta}{\sin\theta\cos\theta}$.

4. $\displaystyle\int\frac{dx}{\sin x+\cos x}$. **5.** $\displaystyle\int\frac{dx}{5+3\cos x}$. **6.** $\displaystyle\int\frac{dx}{3+5\cos x}$.

7. $\displaystyle\int_1^2\frac{dx}{\sqrt{(1+x^2)}}$. **8.** $\displaystyle\int_0^1\sqrt{(1+x^2)}\,dx$. **9.** $\displaystyle\int_0^\pi\frac{dx}{3+2\cos x}$.

10. $\displaystyle\int(1+x^2)^{\frac{3}{2}}\,dx$. **11.** $\displaystyle\int\sqrt{(x^2-16)}\,dx$. **12.** $\displaystyle\int\sqrt{(x^2-3x-10)}\,dx$.

13. $\displaystyle\int_0^{\frac{\pi}{2}}\sin^5\theta\,d\theta$. **14.** $\displaystyle\int_0^{\frac{\pi}{2}}\cos^6\theta\,d\theta$. **15.** $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{10}\theta\cos^3\theta\,d\theta$.

16. $\displaystyle\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^2\theta\,d\theta$. **17.** $\displaystyle\int_0^1(1-x^2)^{\frac{5}{2}}\,dx$.

18. $\displaystyle\int_0^1 x^4(1-x^2)^3\,dx$. **19.** $\displaystyle\int_0^1 x^3(1-x)^4\,dx$.

20. $\displaystyle\int_0^a x^2\sqrt{(a^2-x^2)}\,dx$. **21.** $\displaystyle\int_0^2 x\sqrt{(2x-x^2)}\,dx$.

22. Find the area enclosed by the curve $y^2 = x^3(1-x)$.

23. Sketch the curve given by $x = a\sin^3\theta$, $y = a\sin^2\theta\cos\theta$, and find the area of the portion for which x is positive.

24. Evaluate $\displaystyle\int_0^1 \frac{x^4\,dx}{\sqrt{(1-x^2)}}$.

25. Find the area enclosed by the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

EXAMPLES XIV i

Miscellaneous Examples on Integration

Find the integrals of the expressions in Examples 1—50:

1. $x^{-1\cdot 37}$. **2.** $\sqrt[3]{(a+x)}$. **3.** $\dfrac{1}{1+x^2}$.

4. $\dfrac{x}{(1+2x)^{\frac{1}{3}}}$. **5.** $\dfrac{x}{\sqrt{(a^2-x^2)}}$. **6.** $\dfrac{1}{c-x}$.

7. $\dfrac{x^2}{1-2x^3}$. **8.** $\dfrac{x+a}{x^2+a^2}$. **9.** $\dfrac{\sin x}{a+b\cos x}$.

10. $\dfrac{1}{(a+bx)^2}$. **11.** $\dfrac{x}{(1-3x)^2}$. **12.** $\dfrac{1}{x^2(1+x)}$.

13. $\dfrac{x}{\sqrt{(2+3x)}}$. **14.** $\dfrac{\sqrt{(1+\log x)}}{x}$. **15.** $\dfrac{1}{x\log a}$.

16. $\sin^2 x\cos^2 x$. **17.** $\cot x$. **18.** $\operatorname{cosec} 3x$.

19. $\operatorname{cosec}^2 3x$. **20.** $\dfrac{x^3}{(2+x)^2}$. **21.** $\sin ax \sin bx$.

22. $\dfrac{5x^3+1}{(x-1)(x-2)}$. **23.** $\dfrac{x}{x^2+2x-3}$. **24.** $\dfrac{x}{(1+x)(1+x^2)}$.

25. $\sin^2 3x$. **26.** $x\log x$. **27.** $\cos^{-1} x$.

28. $\sec x$. **29.** $\sin^3 2x\cos 2x$. **30.** $\dfrac{\sqrt{(x^2-a^2)}}{x}$.

31. $x(3-2x)^{\frac{2}{3}}$. **32.** $\dfrac{x^2-5}{2\sqrt{x}}$. **33.** $\dfrac{\log(ax)}{x}$.

34. $\dfrac{\cos x}{1+\sin x}$. **35.** x^2e^x. **36.** $\dfrac{\sin(\log x)}{x}$.

37. $\dfrac{\log(x^2)}{x}$. **38.** $\dfrac{1}{x(1+\log x)}$. **39.** $\dfrac{\tan^{-1} x}{1+x^2}$.

40. $\sqrt{(e^x+4)}$. **41.** $\tan^2(ax+b)$. **42.** 7^x.

43. $\log_2 x$. **44.** $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$. **45.** $\dfrac{x}{\sqrt{(x-1)}}$.

46. $\tan^3 x \sec^2 x$. **47.** $e^{-x}\sin 2x$. **48.** $\tan^3 x$.

49. $\dfrac{1}{\cos^2 x+4\sin^2 x}$. **50.** $\dfrac{1}{x\sqrt{(a^2-x^2)}}$.

51. Prove that

$$\int_0^x \tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int_0^x \tan^{n-2}x\,dx,$$

and evaluate
$$\int_0^{\frac{\pi}{4}} \tan^4 x\,dx.$$

Evaluate the definite integrals in Examples 52—63:

52. $\int_0^\infty e^{-2x}\,dx$. **53.** $\int_0^1 x^2(1-x)^3\,dx$. **54.** $\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta$.

55. $\int_0^\pi \sin^2 10x\,dx$. **56.** $\int_0^{\frac{\pi}{4}} x\sin x\,dx$. **57.** $\int_0^\infty x^2e^{-3x}\,dx$.

58. $\int_0^{\frac{\pi}{2}} \dfrac{dx}{1+\cos x}$. **59.** $\int_0^1 (1-x^2)^{\frac{7}{2}}\,dx$. **60.** $\int_0^1 \dfrac{x\,dx}{\sqrt{(1-x^2)}}$.

61. $\int_0^1 \dfrac{x+1}{(x+2)(x+3)}\,dx$. **62.** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta\,\mathrm{cosec}\,\theta\,d\theta$.

63. $\int_0^{\frac{\pi}{2}} x\sin x\cos x\,dx$.

64. Two long coaxal cylinders of radii a and b are charged to form a condenser. If the potential of the outer one is zero, then the potential V of the inner is given by

$$V=2\int_0^\infty\left[\frac{2\pi a\sigma\,dz}{\sqrt{(a^2+z^2)}}-\frac{2\pi a\sigma\,dz}{\sqrt{(b^2+z^2)}}\right]$$

where σ is a constant: prove that

$$V=4\pi a\sigma\log\left(\frac{b}{a}\right).$$

65. Prove that

$$\int_0^1 x \log (1+\tfrac{1}{2}x)\, dx = \tfrac{3}{4}(1-2\log \tfrac{3}{2}).$$

66. Prove that $$\int_2^{\frac{5}{2}} \sqrt{(x^2-4)}\, dx = \tfrac{15}{8} - 2\log 2.$$

67. Evaluate $$\int_0^{\frac{\pi}{2}} \cos^2 x \cos 2x\, dx.$$

68. Find the area of the loop of the curve $y^2 = x(x-1)^2$.

69. The curve $y = 1 + \sin x$ is rotated about the x-axis; prove that the volume contained between the surface and the planes $x=0$ and $x=\pi$ is

$$\pi\left(4 + \frac{3\pi}{2}\right).$$

70. Prove that the area of the curve $x^4 - x^2 + y^2 = 0$ is $\frac{4}{3}$.

CHAPTER XV

APPLICATIONS TO GEOMETRY

I. The Tangent to a curve in Polar Coordinates

IF O is a fixed point in a fixed straight line OX, called the *initial line*, the position of any point P in a fixed plane through OX is determined when the length r of OP and the angle θ which OP makes with OX are given: (r, θ) are called the *polar coordinates* of P; r is called the *radius vector* and θ is called the *vectorial angle.*

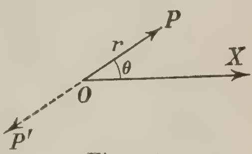

Fig. 157.

It is usual to measure θ as positive for an anti-clockwise rotation from OX and negative for a clockwise rotation: a negative value of r is represented by producing the radius vector backwards through O. Thus if in Fig. 157

$$OP = OP' = 2 \text{ and } \angle XOP = \frac{\pi}{6},$$

the point P could be represented in any of the following ways:

$$\left(2, \frac{\pi}{6}\right), \left(-2, \frac{7\pi}{6}\right), \left(-2, -\frac{5\pi}{6}\right), \left(2, -\frac{11\pi}{6}\right)$$

and P' by $$\left(-2, \frac{\pi}{6}\right), \left(2, -\frac{5\pi}{6}\right) \text{ etc.}$$

If the Cartesian coordinates of P are (x, y), we have

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{ and } \left.\begin{array}{l} r = \sqrt{(x^2+y^2)} \\ \tan\theta = \dfrac{y}{x} \end{array}\right\}.$$

The *Polar equation of a curve* is the relation between the polar coordinates r, θ of any point on the curve.

Example 1.

Find the polar equation of a circle if the origin lies on the circumference and the centre lies on the initial line.

Let the radius $= a$, $\therefore$ $OA = 2a$, $OP = r$, $\angle POA = \theta$.

Since $\angle OPA = \frac{\pi}{2}$, $\underline{r = 2a\cos\theta}$: this is the required equation.

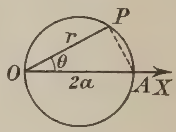

Fig. 158.

Note. When $\pi > \theta > \frac{\pi}{2}$, $\cos\theta$ is negative and $\therefore$ r is negative. The radius vector is $\therefore$ produced backwards to meet the circle at P': see Fig. 159. The whole circumference is described by the variation of θ from 0 to π.

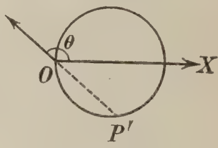

Fig. 159.

Further Notation.

If the tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$ meets OX at T (Fig. 160) and if OY is the perpendicular from O to PT and if A is any fixed point on the curve,

$\angle OPT$ is denoted by ϕ,

OY is denoted by p,

the length of arc AP is denoted by s,

$\angle PTX$ is denoted by ψ.

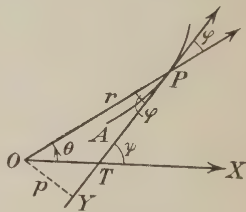

Fig. 160.

We shall now establish the following results:

$$\textbf{(i)}\ \sin\phi = r\frac{d\theta}{ds};\quad \textbf{(ii)}\ \cos\phi = \frac{dr}{ds};$$

$$\textbf{(iii)}\ \tan\phi = r\frac{d\theta}{dr};\quad \textbf{(iv)}\ p = r\sin\phi.$$

In Fig. 161 take the point $Q(r + \delta r, \theta + \delta\theta)$ near Q and draw PN perpendicular to OQ.

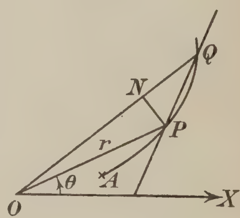

Fig. 161.

Since arc $AP = s$, arc $AQ = s + \delta s$ and arc $PQ = \delta s$.

Now $\angle PON = \delta\theta$,

$\therefore$ $PN = r\sin\delta\theta \doteq r\,\delta\theta$.

Also $$NQ = OQ - ON = (r+\delta r) - r\cos\delta\theta$$
$$= \delta r + r(1-\cos\delta\theta) = \delta r + 2r\sin^2\frac{\delta\theta}{2}$$
$$\doteq \delta r + 2r \times \frac{(\delta\theta)^2}{4} \doteq \delta r + \tfrac{1}{2}r(\delta\theta)^2 \doteq \delta r.$$

Also $$\text{chord } PQ \doteq \text{arc } PQ \doteq \delta s.$$

$$\therefore \quad \sin NQP \doteq \frac{r\delta\theta}{\delta s}, \quad \cos NQP \doteq \frac{\delta r}{\delta s}, \quad \tan NQP \doteq \frac{r\delta\theta}{\delta r}.$$

In the limit when $\delta\theta \to 0$, $\angle NQP \to \phi$ and we have

$$\sin\phi = r\frac{d\theta}{ds}, \quad \cos\phi = \frac{dr}{ds}, \quad \tan\phi = r\frac{d\theta}{dr}.$$

Also since $\angle OYP = \frac{\pi}{2}$, we have $p = r\sin\phi$.

EXAMPLES XV a

1. Find ϕ and p in terms of r, θ for the following loci:

(i) $r = a\sin\theta$; (ii) $r\sin\theta = a$;

(iii) $r(1+\sin\theta) = a$; (iv) $r = e^{\theta\cot\alpha}$.

2. Prove that for the curve $r^2 = a^2\cos 2\theta$, the relation between p, r is $r^3 = a^2 p$. [The p, r equation is called the *pedal equation* of the curve.]

3. Find the pedal equation of the parabola $\frac{2a}{r} = 1 + \cos\theta$.

4. Find the pedal equation of the curve $r^n = a^n \sin n\theta$.

5. Find a point on the curve $r^2 = a^2\cos 2\theta$ at which the tangent makes an angle $\frac{3\pi}{4}$ with the initial lines.

6. Sketch $r = a\theta$, the spiral of Archimedes, and find for what value of r, $\phi = \frac{\pi}{4}$.

7. Prove that for any curve $\frac{ds}{d\theta} = \frac{r^2}{p}$.

8. Prove that for any curve $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

II. Areas in polar coordinates

The area bounded by the curve $r=f(\theta)$ and the lines $\theta=\alpha$, $\theta=\beta$ is $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$.

With the notation of Fig. 161, p. 236, let the area of the sector AOP be z.

Then $\delta z =$ area of sector POQ.

$$\therefore\ \tfrac{1}{2}r^2\sin\delta\theta < \delta z < \tfrac{1}{2}(r+\delta r)^2\sin\delta\theta,$$

$$\therefore\ \tfrac{1}{2}r^2\frac{\sin\delta\theta}{\delta\theta} < \frac{\delta z}{\delta\theta} < \tfrac{1}{2}(r+\delta r)^2\frac{\sin\delta\theta}{\delta\theta}.$$

$\therefore$ when $\delta\theta\to 0$ since $\delta r\to 0$ and $\dfrac{\sin\delta\theta}{\delta\theta}\to 1$, we have

$$\frac{dz}{d\theta}=\tfrac{1}{2}r^2.$$

$$\therefore\ z=\tfrac{1}{2}\int r^2\,d\theta + c.$$

But $z=0$ when $\theta=\alpha$.

$\therefore$ area of sector AOP $=\frac{1}{2}\int_\alpha^\theta r^2\,d\theta$

and the required area $=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta.$

Note. If we use the formula, the area of $\Delta\ OPQ=\frac{1}{2}$ height $\times$ base, we have $\delta z \doteqdot \frac{1}{2}p\,.\,\delta s$.

If $s=s_1$ and $s=s_2$ correspond to $\theta=\alpha$ and $\theta=\beta$, we then have area of sector $=\frac{1}{2}\int_{s_1}^{s_2} p\,.\,ds.$

Example 2.

Trace the cardioid $r=a(1+\cos\theta)$ and find the area it encloses.

Draw the circle with diameter $OA=a$, join O to any point Q on the circle and produce OQ to P so that $QP=a$, then the locus of P is the required curve, for

$$OP=OQ+QP=a\cos\theta+a.$$

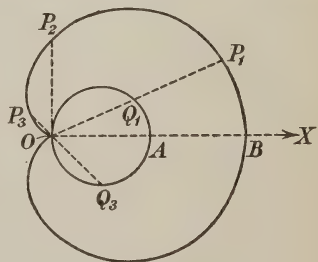

Fig. 162.

When $\theta=\frac{\pi}{2}$, $OP_2=a$; when $\theta=\frac{2\pi}{3}$, $OQ_3=-\frac{a}{2}$, $OP_3=a-\frac{a}{2}=\frac{a}{2}$; when $\theta=\pi$, $OQ=-a$ and $OP=0$ so that P is at O.

The curve is traced out once completely when θ varies from 0 to 2π.

$$\therefore \text{ area } = \tfrac{1}{2}\int_0^{2\pi} r^2\,d\theta = \int_0^{\pi} r^2\,d\theta = a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$$

$$=a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=a^2\int_0^{\pi}\left\{1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right\}d\theta$$

$$=a^2\left[\tfrac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi}=\frac{3\pi a^2}{2}.$$

Alternatively $a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta=4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta.$

Putting $\theta=2\omega$ the integral becomes

$$8a^2\int_0^{\frac{\pi}{2}}\cos^4\omega\,d\omega=8a^2\times\tfrac{3}{4}\times\tfrac{1}{2}\times\frac{\pi}{2}=\frac{3\pi a^2}{2}.$$

(See pp. 229—230.)

Example 3.

Prove that the tangent to the cardioid $r=a(1+\cos\theta)$ at the point (r, θ) makes an angle $\frac{\pi}{2}+\frac{\theta}{2}$ with the radius vector.

$$r=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}.$$

$$\therefore \log r=\log 2a+2\log\cos\frac{\theta}{2}, \quad \therefore \frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2}.$$

But $\tan\phi=r\dfrac{d\theta}{dr}$, $\therefore \cot\phi=\dfrac{1}{r}\dfrac{dr}{d\theta}=-\tan\dfrac{\theta}{2}=\cot\left(\dfrac{\pi}{2}+\dfrac{\theta}{2}\right)$.

$$\therefore \phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

EXAMPLES XV b

1. Find the area enclosed by $r=a\cos\theta$.

2. Find the area enclosed by $r=a(1-\cos\theta)$.

3. For what values of θ is r^2 negative if $r^2=a^2\cos 2\theta$?

Sketch the graph of $r^2=a^2\cos 2\theta$ and find the area of one loop.

4. Prove that for $r\theta=a$ the area from (r_1, θ_1) to (r_2, θ_2) is proportional to r_1-r_2.

5. Find the area between the hyperbola $r^2 \sin 2\theta=2c^2$ and the lines

$$\theta=\frac{\pi}{6}, \quad \theta=\frac{\pi}{3}.$$

6. Trace the curve $r=1+2\cos\theta$ and show that it consists of two loops. What area is obtained by integrating $\frac{1}{2}r^2\,d\theta$ from 0 to 2π? What is the relation between r and θ for the inner loop? What limits of integration for θ lead to the area of the inner loop?

7. Using the relations $r^2=x^2+y^2$, $\tan\theta=\dfrac{y}{x}$,

prove that $$\tfrac{1}{2}\int r^2\,d\theta=\tfrac{1}{2}\int (x\,dy-y\,dx).$$

8. The polar equation of an ellipse with its centre O as origin and its semi-major axis OA as initial line is $r^2\left(\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}\right)=1$; P is a point on the curve such that $\angle AOP=\alpha$; find the area of the sector AOP.

III. The Centroid in Polar Coordinates

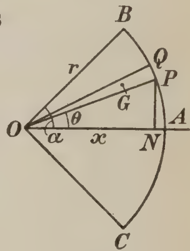

Fig. 163.

Find the centroid of (i) a circular arc, (ii) a circular sector.

(i) If 2α is the angle subtended by the arc at the centre, take the bisector as the initial line. Then if $AP=s$,

$$PQ=\delta s=r\delta\theta, \quad x=ON$$

and $$\text{arc } CAB=2r\,.\,\alpha.$$

Then $$\bar{x} \doteq \frac{\Sigma\,\delta s\,.\,x}{\Sigma\,\delta s}=\frac{\int_{-\alpha}^{+\alpha}(r\,d\theta)(r\cos\theta)}{\int_{-\alpha}^{+\alpha} r\,d\theta}$$

$$=\frac{r^2\int_{-\alpha}^{+\alpha}\cos\theta\,d\theta}{\int_{-\alpha}^{+\alpha} r\,d\theta}=\frac{r^2\Big[\sin\theta\Big]_{-\alpha}^{+\alpha}}{r\Big[\theta\Big]_{-\alpha}^{+\alpha}}=\frac{r^2\,.\,2\sin\alpha}{r\,.\,2\alpha}$$

$$=\frac{r\sin\alpha}{\alpha}.$$

(ii) The centroid of the element POQ will be at G whose abscissa is approximately $\frac{2}{3} r \cos\theta$;

$$\therefore \bar{x} \doteqdot \frac{\Sigma\left(\frac{1}{2} r^2 \delta\theta\right)\left(\frac{2}{3} r \cos\theta\right)}{\Sigma\left(\frac{1}{2} r^2 \delta\theta\right)},$$

$$\bar{x} = \frac{\frac{1}{3}\int_{-\alpha}^{+\alpha} r^3 \cos\theta \, d\theta}{\frac{1}{2}\int_{-\alpha}^{+\alpha} r^2 \, d\theta} = \frac{2}{3} r \frac{\sin\alpha}{\alpha}.$$

EXAMPLES XV c

1. Find the centre of gravity of a wire in the form of a semicircular arc.

2. Find the centre of gravity of a semicircular area.

3. P is any point on the semicircle on OA as diameter; the equation of the circle with origin O and initial line OA is $r = 2a\cos\theta$, where a is the radius; show that the distance from OA of the centre of gravity of the area AOP is $\frac{2a}{3} \cdot \frac{1-\cos^4\alpha}{\alpha + \sin\alpha\cos\alpha}$, where $\angle AOP = \alpha$.

4. Prove that the centre of gravity of the area bounded by the cardioid $r = a(1+\cos\theta)$ lies on the initial line at a distance $\frac{5a}{6}$ from the origin.

5. Find the distance from the origin of the centre of gravity of one loop of $r^2 = a^2 \cos 2\theta$.

6. If O is the origin and if the initial line cuts the cardioid

$$r = a(1+\cos\theta)$$

at A and if $P(r, \theta)$ is any point on the curve, it can be proved that the length of the arc $AP = 4a\sin\frac{\theta}{2}$; use this result to find the centre of gravity of a wire in the form of a curve $r = a(1+\cos\theta)$.

IV. Length of an arc

(i) *Cartesian Coordinates.*

P, Q are the points (x, y), $(x+\delta x, y+\delta y)$ on the curve $y = f(x)$, see Fig. 164.

A is any fixed point on the curve, and the length of arc $\overset{\frown}{AP} = s$, $\overset{\frown}{PQ} = \delta s$.

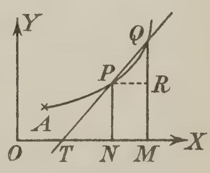

Fig. 164.

Now $\overline{PQ}^2 = PR^2 + RQ^2$, where $\overline{PQ}$ is the chord PQ,

$$\text{i.e. } \overline{PQ}^2 = \delta x^2 + \delta y^2.$$

$$\therefore \left.\overline{\frac{\overline{PQ}}{\delta x}}\right|^2 = 1 + \left.\overline{\frac{\delta y}{\delta x}}\right|^2$$

which may be written

$$\left(\frac{\overline{PQ}}{\widehat{PQ}} \times \frac{\widehat{PQ}}{\delta x}\right)^2 = 1 + \left.\overline{\frac{\delta y}{\delta x}}\right|^2.$$

When $\delta x \rightarrow 0$, the limit of $\dfrac{\overline{PQ}}{\widehat{PQ}} = 1$ and $\underset{\delta x \rightarrow 0}{\text{Lt}} \dfrac{\delta s}{\delta x} = \dfrac{ds}{dx}$.

$$\therefore \left.\overline{\frac{ds}{dx}}\right|^2 = 1 + \left.\overline{\frac{dy}{dx}}\right|^2,$$

$$\text{i.e. } \frac{ds}{dx} = \sqrt{\left\{1 + \left.\overline{\frac{dy}{dx}}\right|^2\right\}} \text{ and } s = \int \sqrt{\left(1 + \left.\overline{\frac{dy}{dx}}\right|^2\right)}\, dx + c.$$

$\therefore$ the length of arc from $(x_1\, y_1)$ to $(x_2\, y_2)$ is

$$\int_{x_1}^{x_2} \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx.$$

For most curves, this integral cannot be expressed in terms of elementary functions.

If the curve is given by $x = f(t)$, $y = \theta(t)$ by using the relation $(\delta s)^2 \doteq (\delta x)^2 + (\delta y)^2$ we have $\left(\dfrac{ds}{dt}\right)^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$ from which s is found by integration.

If the chord QP meets OX at T (Fig. 164), we have $\cos PTN = \cos QPR = \dfrac{PR}{PQ} = \dfrac{\delta x}{\delta s} \times \dfrac{\delta s}{\overline{PQ}}$.

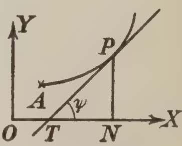

Fig. 165.

If the tangent at P makes an angle ψ with OX (Fig. 165), then

$$\cos\psi = \text{Lt}\, \frac{\delta x}{\delta s} \times \frac{\delta s}{\overline{PQ}} = \frac{dx}{ds}.$$

Similarly $\sin\psi = \text{Lt}\, \dfrac{\delta y}{\delta s} \times \dfrac{\delta s}{\overline{PQ}} = \dfrac{dy}{ds}$ and as usual $\tan\psi = \dfrac{dy}{dx}$.

From these results, since $\sec^2\psi = 1 + \tan^2\psi$, we have

$$\overline{\frac{ds}{dx}}\Big|^2 = 1 + \overline{\frac{dy}{dx}}\Big|^2$$

as before.

Example 4.

Find the length of the curve $y = \sqrt{a^2 - x^2}$ from $x = 0$ to $x = \frac{a}{2}$.

$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}, \quad \therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

$$\therefore \frac{ds}{dx} = \frac{a}{\sqrt{(a^2 - x^2)}} \text{ or } s = \int \frac{a\,dx}{\sqrt{(a^2 - x^2)}}.$$

$$\therefore \text{length of arc required} = \int_0^{\frac{a}{2}} \frac{a\,dx}{\sqrt{(a^2 - x^2)}} = a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{\frac{a}{2}}$$

$$= a\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = a \times \frac{\pi}{6} = \frac{\pi a}{6}.$$

Note. The equation $y = \sqrt{(a^2 - x^2)}$ or $y^2 + x^2 = a^2$ represents a circle radius a, centre the origin: and the required result is of course obtained at once from elementary trigonometry.

(ii) *Polar Coordinates.*

With the notation of p. 236, we have

$$\tan\phi = r\frac{d\theta}{dr}, \quad \cos\phi = \frac{dr}{ds}.$$

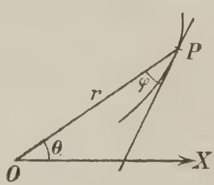

Fig. 166.

But

$$\sec^2\phi = 1 + \tan^2\phi, \quad \therefore \left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2;$$

$$\therefore \frac{ds}{dr} = \sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}} \text{ and } s = \int\sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}}\,dr.$$

This integral cannot usually be expressed in terms of elementary functions: and even when this can be done, it is often better first of all to find ϕ in terms of r, θ and then to integrate either

$$\frac{ds}{dr} = \sec\phi \quad \text{or} \quad \frac{1}{r}\frac{ds}{d\theta} = \operatorname{cosec}\phi.$$

Example 5.

Find the length of the cardioid $r=a(1+\cos\theta)$.

The graph is given on p. 238.

We have $r=2a\cos^2\frac{\theta}{2}$, $\therefore \log r=\log(2a)+2\log\left(\cos\frac{\theta}{2}\right)$.

$$\therefore \frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2} \text{ or } \cot\phi=-\tan\frac{\theta}{2}=\cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right);$$

$$\therefore \phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

$$\therefore \frac{ds}{d\theta}=r\operatorname{cosec}\phi=2a\cos^2\frac{\theta}{2}\operatorname{cosec}\left(\frac{\pi}{2}+\frac{\theta}{2}\right)=2a\cos^2\frac{\theta}{2}\sec\frac{\theta}{2}=2a\cos\frac{\theta}{2};$$

$$\therefore s=\int 2a\cos\frac{\theta}{2}\,d\theta+c$$

$$=4a\sin\frac{\theta}{2}+c.$$

If we measure $s=0$ from $\theta=0$ we have $c=0$;

$$\therefore s=4a\sin\frac{\theta}{2}.$$

$\therefore$ the length of the portion above OX is obtained by putting $\theta=\pi$.

$\therefore$ total length of curve $=2\times 4a\sin\frac{\pi}{2}=8a$.

EXAMPLES XV d

1. Find the length of the arc of $x^3=y^2$ from $x=0$ to $x=1$.

2. For the curve $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, prove that

$$\frac{ds}{d\theta}=2a\cos\frac{\theta}{2}$$

and calculate the length of the portion of the curve (a cycloid) between $\theta=0$ and $\theta=\pi$.

3. Find the total length of the curve $r=a(1-\cos\theta)$.

4. Find the length of the arc of $x^3=8y^2$ from $x=0$ to $x=2$.

5. Find the length of the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$.

6. Find the length of the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$ from $x=0$ to $x=a$.

7. Show that for the parabola $y^2=4ax$, $\frac{ds}{dx}=\sqrt{\left(1+\frac{a}{x}\right)}$.

8. Show that for the parabola $y=ax^2$, $\frac{ds}{dx}=\sqrt{(1+4a^2x^2)}$: if x is small, $\frac{ds}{dx} \doteqdot 1+2a^2x^2$; use this result to find the length of arc from (0, 0) to (h, k) if h is small.

9. Find the length of arc cut off between $\theta=\alpha$ and $\theta=\beta$ from $r=a.e^{\theta\cot\gamma}$.

10. For the curve $r\cos^3\theta=a\sin^2\theta$, prove that

$$\frac{ds}{d\theta}=a\tan\theta\sec^2\theta\,(4+9\tan^2\theta)^{\frac{1}{2}}$$

and show that the length of the portion of the curve between $\theta=0$ and $\theta=\tan^{-1}\left(\frac{\sqrt{5}}{3}\right)$ is $\frac{19a}{27}$.

V. Curvature

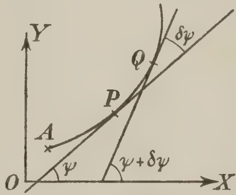

Fig. 167.

A is a fixed point on the curve $y=f(x)$; the tangents at

$P(x, y)$ and $Q(x+\delta x, y+\delta y)$

make angles ψ, $\psi+\delta\psi$ with OX; and the arcs AP, AQ are of lengths s, $s+\delta s$.

The angle between the tangents at P, Q is $\delta\psi$ and arc $PQ=\delta s$.

$\therefore$ $\frac{\delta\psi}{\delta s}$ measures the average change in direction per unit increase of arc for the arc PQ (i.e. the rate at which the curve bends from P to Q) and is called the *average curvature* of the arc PQ.

When $\delta s \to 0$, Lt $\frac{\delta\psi}{\delta s}=\frac{d\psi}{ds}$ and this is called the *curvature* of the curve at P.

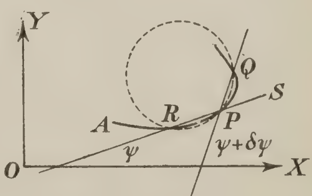

Fig. 168.

Consider three points P, Q, R on the curve (Fig. 168) given by

arc $AP=s$, arc $AQ=s+\delta s$,

arc $AR=s-\delta s$

and draw a circle through P, Q, R: then its radius

$$=\frac{\overline{RQ}}{2\sin QPR} \doteqdot \frac{\text{arc } RPQ}{2\sin QPS} \doteqdot \frac{2\delta s}{2\sin QPS}.$$

Therefore when $\delta s \to 0$, the radius of the limiting circle

$$=\rho=\text{Lt}\,\frac{\delta s}{\sin QPS}=\text{Lt}\,\frac{\delta s}{\delta\psi}=\frac{ds}{d\psi}.$$

This limiting circle is called the *circle of curvature* at P; it touches the curve at P and its curvature is equal to that of the curve. The radius of this circle, which equals $\frac{ds}{d\psi}$, is called the *radius of curvature* at P and equals the reciprocal of the curvature.

For a circle, radius a (see Fig. 169), we have $s=a\psi$ and $\frac{ds}{d\psi}=a$. Also the curvature

$$=\frac{d\psi}{ds}=\frac{1}{a}.$$

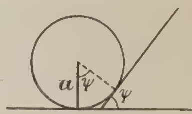

Fig. 169.

The circle is the only plane curve of constant curvature.

Note. The circle of curvature at any point P of a curve is the circle which touches the curve at P and which bends round at the same rate as the curve is bending at P.

(i) The radius of curvature at the point (x, y) on $y=f(x)$ is given by

$$\rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

We have
$$\frac{dy}{dx}=\tan\psi;$$

$$\therefore\ \frac{d^2y}{dx^2}=\sec^2\psi\,\frac{d\psi}{dx}=\sec^2\psi\,\frac{d\psi}{ds}\cdot\frac{ds}{dx}=\sec^3\psi\,\frac{d\psi}{ds}\text{ since }\frac{ds}{dx}=\sec\psi;$$

$$\therefore\ \frac{d^2y}{dx^2}=(\sec^2\psi)^{\frac{3}{2}}\,\frac{d\psi}{ds}=(1+\tan^2\psi)^{\frac{3}{2}}\,\frac{d\psi}{ds}.$$

$$\therefore\ \rho=\frac{ds}{d\psi}=\frac{(1+\tan^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Note. If $\frac{dy}{dx}$ is small, e.g. along a rod bending slightly under its own weight,
$$\rho \doteq \frac{1}{\frac{d^2y}{dx^2}}.$$

(ii) The radius of curvature at a point on a curve given by its pedal equation
$$r=f(p) \text{ is } \rho=r\frac{dr}{dp}.$$

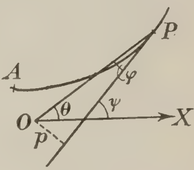

Fig. 170.

With the usual notation
$$\frac{1}{\rho}=\frac{d\psi}{ds}=\frac{d}{ds}(\theta+\phi)=\frac{d\theta}{ds}+\frac{d\phi}{ds}.$$

Now $\frac{p}{r}=\sin\phi$, $\therefore \frac{1}{r}\frac{dp}{dr}-\frac{p}{r^2}=\cos\phi\frac{d\phi}{dr}$;
$$\therefore \frac{1}{r}\frac{dp}{dr}=\frac{r\sin\phi}{r^2}+\frac{dr}{ds}\cdot\frac{d\phi}{dr}=\frac{1}{r}\cdot\frac{rd\theta}{ds}+\frac{d\phi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}=\frac{1}{\rho};$$
$$\therefore \rho=r\frac{dr}{dp}.$$

Note. If we wish to find the radius of curvature for a curve given in polar coordinates, it is best to start by finding the pedal equation and then use $\rho=r\frac{dr}{dp}$.

Example 6.

Find the radius of curvature at the point (r, θ) on the parabola
$$\frac{2a}{r}=1+\cos\theta.$$

We have $\qquad \frac{a}{r}=\cos^2\frac{\theta}{2}$ or $\log a-\log r=2\log\cos\frac{\theta}{2}$;
$$\therefore -\frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2};$$
$$\therefore \cot\phi=\tan\frac{\theta}{2}=\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \text{ or } \phi=\frac{\pi}{2}-\frac{\theta}{2};$$
$$\therefore p=r\sin\phi=r\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=r\cos\frac{\theta}{2};$$
$$\therefore p^2=r^2\cos^2\frac{\theta}{2}=r^2\times\frac{a}{r}=ar;$$
$$\therefore 2p\frac{dp}{dr}=a;$$
$$\therefore \rho=r\frac{dr}{dp}=r\times\frac{2p}{a}=\frac{2r}{a}\times\sqrt{ar}=2\sqrt{\left(\frac{r^3}{a}\right)}.$$

EXAMPLES XV e

1. Find the radius of curvature at (1, 1) on $y^2=x^3$.

2. Find the radius of curvature at the origin on

$$\text{(i) } y=ax^2, \quad \text{(ii) } y=ax^2+bx^3.$$

3. Find the radius of curvature at the point (x, y) on

(i) the hyperbola $xy=c^2$,

(ii) the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$.

4. Find the radius of curvature at the point (r, θ) on

(i) the equiangular spiral $r=a\,e^{\theta\cot\alpha}$.

(ii) the cardioid $r=a\,(1+\cos\theta)$.

5. If in the cardioid $r=a\,(1+\cos\theta)$, s is measured from $\theta=0$ and if $0<\theta<\pi$, prove that $s^2+9\rho^2=16a^2$.

6. Find the radius of curvature at the point θ on the cycloid given by $x=a\,(\theta+\sin\theta)$, $y=a\,(1-\cos\theta)$.

7. Find the radius of curvature at the point t on the curve

$$x=a\cos^3 t, \quad y=a\sin^3 t.$$

8. If the curve $y=f(x)$ touches the x-axis at the origin, prove that the radius of curvature at the origin $=\underset{x\to 0}{\mathrm{Lt}}\,\dfrac{x^2}{2y}$.

9. If $\dfrac{d^2y}{dx^2}$ vanishes and changes sign in passing through the point P on the curve $y=f(x)$, what geometrical interpretation is suggested by the formula for ρ ?

10. If OY is the perpendicular from the origin O to the tangent at any point P on the curve, prove with the usual notation that $PY=\dfrac{dp}{d\psi}$.

$$\left[\text{Note } \rho=\frac{ds}{d\psi}=r\,\frac{dr}{dp}.\right]$$

11. Prove that the radius of curvature at the point (x, y) on the parabola $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{y}{b}}=1$ is $\dfrac{2}{ab}(ax+by)^{\frac{3}{2}}$.

12. Find the radius of curvature at the point $(a\cos\phi, b\sin\phi)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

VI. Surface of a Solid of Revolution

When the arc AB rotates about OX and traces out the surface of a solid of revolution, the chord PQ of the small element δs, where arc $AP=s$, will trace out the surface of the frustum of a cone whose area is given by (circumference of mean section) × (slant height).

$\therefore$ the chord PQ traces out a surface of area δS where

$$\delta S = 2\pi \left(\frac{y+(y+\delta y)}{2}\right)\overline{PQ},$$

S being the area traced out by the arc AP.

$$\therefore \frac{\delta S}{\overline{PQ}} = \pi\,[2y+\delta y].$$

$$\therefore \frac{\delta S}{\widehat{PQ}} \times \frac{\widehat{PQ}}{\overline{PQ}} = \pi\,[2y+\delta y],$$

$$\therefore \frac{\delta S}{\delta s} \doteq 2\pi y,$$

and in the limit when $\delta s \to 0$

$$\frac{dS}{ds} = 2\pi y, \quad \therefore\ \underline{S = \int 2\pi y\, ds.}$$

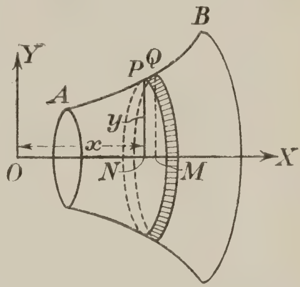

Fig. 171.

It is important to notice that this formula is *not* $\int 2\pi y\,dx$: the area is *not* traced out by an element δx but by δs.

If the curve is given in polar coordinates and if the axis of rotation is the initial line, this formula may be written

$$S = 2\pi \int r \sin\theta\, ds.$$

Example 7.

Find the area of a spherical cap of height h. Let $AB=h$, and let R be the radius of the sphere; then if the coordinates of P are (x, y), we have $x^2+y^2=R^2$.

$$\therefore 2x\,dx+2y\,dy=0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \frac{ds}{dx} = \sqrt{1+\left|\frac{dy}{dx}\right|^2} = \sqrt{1+\frac{x^2}{y^2}} = \frac{R}{y}.$$

$$\therefore y\,ds = R\,dx.$$

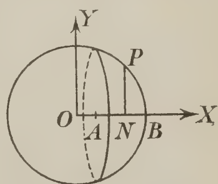

Fig. 172.

$$\therefore\ S = \int 2\pi y\,ds = \int_{R-h}^{R} 2\pi R\,dx$$

$$= 2\pi R \int_{R-h}^{R} dx = 2\pi R \Big[\, x \,\Big]_{R-h}^{R}$$

$$= \underline{2\pi R h.}$$

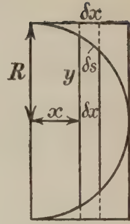

Fig. 173.

In particular, the area of the surface of the whole sphere

$$= 2\pi R\,(2R) = 4\pi R^2.$$

Note that the relation $y\,ds = R\,dx$ shows that

$$2\pi y\,ds = 2\pi R\,dx,$$

i.e. the area of the belt of surface of sphere = area of corresponding belt of the circumscribing cylinder. [Archimedes' Theorem.]

Theorems of Pappus and Guldin

(1) If a plane area A revolves about an axis lying in its plane but not intersecting it, the volume of the solid so generated is measured by

$A \times$ length of path of centroid of area.

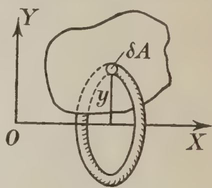

Fig. 174.

Consider a small element δA at a distance y from the axis OX.

Volume of ring traced out by $\delta A \doteq 2\pi y\,\delta A$.

$\therefore$ volume of whole solid $\doteq 2\pi \Sigma y\,\delta A$.

But if $\bar{y}$ is the distance from OX of the centroid of A, we have

$$\bar{y} \doteq \frac{\Sigma\,(y\,\delta A)}{\Sigma\,\delta A} \doteq \frac{\Sigma\,(y\,\delta A)}{A}.$$

$$\therefore\ \Sigma\,(y\,\delta A) \doteq \bar{y} \times A.$$

$\therefore$ when $\delta A \to 0$ we have

volume of whole solid $= 2\pi \int y\,dA$

and $\int y\,dA = \bar{y} \times A$.

$\therefore$ volume of whole solid $= 2\pi \bar{y} \times A$

$= A \times$ length of path of centroid.

Example 8.

Find the volume of an anchor ring formed by the rotation of a circle of radius a about a line at distance d from the centre, where $d > a$.

The centroid traces out a circle of radius d.

$\therefore$ the volume of the anchor ring $= \pi a^2 \times 2\pi d$

$$= 2\pi^2 a^2 d.$$

A simple example of an anchor ring or *tore*, as it is also called, is the ring on a curtain-pole.

(2) When an arc CD of a plane curve of length S revolves about an axis lying in its plane but not intersecting it, the area of the surface so generated is measured by

$S \times$ length of path of centroid of arc.

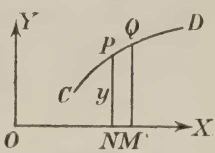

Fig. 175.

Consider an element $PQ = \delta s$ of CD which is rotated about the axis OX and let the ordinate $PN = y$.

Then it is shown on p. 249 that the area of the surface traced out by $PQ \doteq 2\pi y\,\delta s$ and the area traced out by $CD = \int 2\pi y\,ds = 2\pi \int y\,ds$.

Also if the centroid of arc CD is at a distance $\bar{y}$ from OX, we have

$$\bar{y} \doteq \frac{\Sigma\,(y\,\delta s)}{\Sigma\,\delta s} \text{ and } \bar{y} = \frac{\int y\,ds}{S} \text{ or } \bar{y} \times S = \int y\,ds.$$

$\therefore$ the area traced out by $CD = 2\pi\bar{y} \times S$

$= S \times$ length of path of centroid of arc.

Example 9.

Find the area of the surface of an anchor ring formed by the rotation of a circle of radius a about a line at distance d from the centre, where $d > a$.

The centroid traces out a circle of radius d.

$\therefore$ area of surface of anchor ring $= 2\pi a \times 2\pi d$

$$= 4\pi^2 ad.$$

Note. (i) The statements in (1) and (2) also hold good for a rotation through any angle less than four right angles.

(ii) If the expressions for the volume or the area of a surface of revolution are known, these theorems enable us to find the distance of the centroid from the axis of rotation of the area or curve from which they have been generated.

EXAMPLES XV f

1. Find the curved surface of a right cone, base-radius r inches, slant side l inches.

2. Find the surface generated by the parabola $y^2=4ax$ when the portion from (0, 0) to (x, y) revolves about OX.

3. Find the surface of a reflector whose shape is a paraboloid of depth 10 inches and greatest diameter 2 feet 6 inches.

4. Show that, if ϕ varies, the point $x=a\sin^3\phi$, $y=a\cos^3\phi$ traces out the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. Hence find the area of the surface generated by rotating this curve about the x-axis.

5. For the cardioid $r=a(1+\cos\theta)$, prove that $\frac{ds}{dr}=-\operatorname{cosec}\frac{\theta}{2}$; hence find the area of the surface formed by rotating the curve about the initial line.

6. For the "figure-of-eight" curve $r^2=a^2\cos 2\theta$, prove that

$$\frac{ds}{dr}=-\operatorname{cosec} 2\theta;$$

hence find the area of the surface formed by rotating the curve about the initial line.

7. If the portion of the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$ between $x=0$ and $x=a$ is rotated (i) about the x-axis, (ii) about the y-axis, find the area of the surface generated.

8. Interpret geometrically the expression $\int 2\pi(y\cos a-x\sin a)\,ds$, where a is a constant and the integral is taken along a curve of which δs is an element.

9. A curtain ring has an external diameter of 3 inches and its cross-section is a circle of diameter $\frac{1}{2}$ inch. Find the area of its surface.

10. The triangle ABC right-angled at B is rotated about BC to generate a circular cone; $BC=h$, $AB=r$; deduce from Pappus' theorem the volume and area of the surface of the cone.

11. (i) By assuming the surface of a sphere $=4\pi r^2$, find by Pappus' theorem the position of the centre of gravity of a semicircular arc.

(ii) A semicircle of radius $\frac{1}{2}$ inch is placed with its diameter parallel to OX and 1 inch away from it and its rim on the side of the diameter remote from OX; find the area of the surface generated by rotating it about OX.

12. (i) Deduce from Archimedes' expression for the area of a spherical cap (p. 250) the position of the centre of gravity of an arc of a circle radius a subtending an angle 2α at the centre.

(ii) An arc of a circle of radius r is of length $2r\alpha$; find the area of the surface generated by rotating the arc about its chord.

13. By assuming the volume of a sphere $= \frac{4}{3}\pi r^3$, deduce the position of the centroid of a semicircular lamina.

14. A hemispherical bowl of radius 2 feet contains water; find the depth of the water when half the surface is wetted.

15. AB is the diameter of a semicircle, centre O; C is the mid-point of arc AB; OC is produced to D so that $OC = CD = 2$ inches. The semicircular arc is rotated about the line through D parallel to AB; find the area of the surface so obtained.

VII. **Two Important Curves**

(i) *The Catenary.*

A catenary is the curve assumed by a uniform flexible chain when suspended from two points.

With the notation of Fig. 176, we see that the portion CP of the chain is in equilibrium under the action of the tension T at P, the horizontal tension T_0 at C and the weight ws, where w is the weight of unit length.

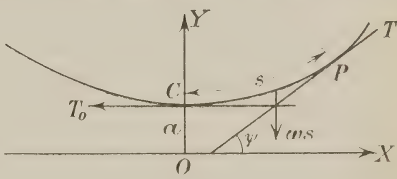

Fig. 176.

Resolving horizontally and vertically,

$$T\cos\psi = T_0 \text{ and } T\sin\psi = ws. \quad \therefore \tan\psi = \frac{ws}{T_0}.$$

Put $T_0 = wa$, i.e. suppose the tension at C to be the weight of a length a of chain: then $\frac{s}{a} = \tan\psi$ or $s = a\tan\psi$.

We wish to deduce the Cartesian equation from this result.

We have $\frac{ds}{dy} = a\sec^2\psi\frac{d\psi}{dy}$ but $\frac{ds}{dy} = \operatorname{cosec}\psi$. $\therefore \frac{dy}{d\psi} = a\frac{\sin\psi}{\cos^2\psi}$.

$$\therefore\ y = a\int \frac{\sin\psi}{\cos^2\psi}\,d\psi = a\sec\psi,$$

if we choose $y=a$ when $\psi=0$. Similarly

$$\frac{ds}{dx} = a\sec^2\psi\,\frac{d\psi}{dx} \text{ but } \frac{ds}{dx} = \sec\psi.$$

$$\therefore\ \frac{dx}{d\psi} = a\sec\psi,$$

and
$$x = a\int\frac{d\psi}{\cos\psi} = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right),$$

see p. 227, if $x=0$ when $\psi=0$.

$$\therefore\ e^{\frac{x}{a}} = \tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right) \text{ and } e^{-\frac{x}{a}} = \cot\left(\frac{\pi}{4}+\frac{\psi}{2}\right).$$

$$\therefore\ e^{\frac{x}{a}} + e^{-\frac{x}{a}} = \frac{\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} + \frac{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} = \frac{1}{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}$$

$$= \frac{2}{\sin\left(\frac{\pi}{2}+\psi\right)} = 2\sec\psi = 2\,\frac{y}{a}.$$

$$\therefore\ y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

Note: the equation of the catenary may be expressed in any of the forms obtained above

(i) $s = a\tan\psi$.

(ii) $y = a\sec\psi$.

(iii) $x = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$.

(iv) $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

From (i) and (ii) we deduce the result

$$s^2 = a^2\tan^2\psi = a^2(\sec^2\psi - 1) = y^2 - a^2.$$

(ii) *The Cycloid.*

When a circle rolls along a straight line, the path traced out by any point P on its circumference is called a *cycloid*.

Let C be the centre and a the radius of the circle which is rolling along the line OX and let the origin O be the starting-point of P so that arc $MP = MO$.

When the circle has turned through an angle θ (i.e. $\angle MCP = \theta$), let the coordinates of P be x, y so that $ON = x$, $NP = y$.

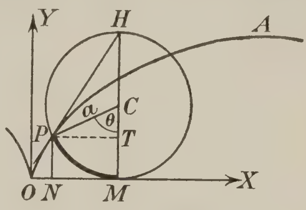

Fig. 177.

Then

$$x = ON = OM - NM = \widehat{PM} - PT = a\theta - a\sin\theta = a(\theta - \sin\theta),$$

and $\quad y = NP = MT = MC - TC = a - a\cos\theta = a(1-\cos\theta).$

The equation of the cycloid is therefore $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$.

Since the circle is rolling without slipping, M is instantaneously at rest and $\therefore$ the circle is instantaneously turning about M.

$\therefore$ P is moving at right angles to PM.

But if MH is a diameter, $\angle MPH$ is a right angle.

$\therefore$ PH is the tangent at P.

This result should be verified by Calculus methods.

The other chief property of the cycloid is as follows:

If A is the vertex of the curve (i.e. the point at which the tangent is parallel to OX), the arc AP = twice the chord HP.

We have

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta = a^2(2-2\cos\theta) = 4a^2\sin^2\frac{\theta}{2}.$$

$$\therefore \frac{ds}{d\theta} = 2a\sin\frac{\theta}{2} \text{ and arc } PA = \int_\theta^\pi 2a\sin\frac{\theta}{2}\,d\theta = 4a\cos\frac{\theta}{2}.$$

But $\qquad \angle PHM = \tfrac{1}{2}\angle PCM = \dfrac{\theta}{2}.$

$$\therefore \text{arc } PA = 4a\cos PHM = 2HM\cos PHM = 2PH.$$

EXAMPLES XV g

The Catenary

1. Show that the tension at any point of the catenary equals the weight of a piece of chain whose length equals the ordinate at that point. If a chain hangs over two smooth pegs, what is the equilibrium position?

2. In Fig. 176, prove that arc $CP=\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$.

3. Show that the area bounded by the x-axis, the ordinates $x=0$ and $x=x_1$ and the catenary is $a\sqrt{(y_1^2-a^2)}$.

4. If in Fig. 176, N is the foot of the perpendicular from P to OX and if Y is the foot of the perpendicular from N to the tangent at P, prove that $NY=a$ and that arc $CP=YP$.

5. If the normal at P meets the x-axis at G, prove that $PG=\frac{y^2}{a}$.

The Cycloid

6. In Fig. 177 prove that the tangent at P makes an angle $\frac{\pi}{2}-\frac{\theta}{2}$ with OX.

7. Prove that the length of the arc of one arch of the cycloid $=8a$.

8. Prove that the area enclosed by OX and one arch $=3\pi a^2$.

9. If s is measured from A and if the initial line is the tangent at A, prove that the equation of the cycloid can be written in the form

$$s=4a\sin\psi.$$

10. Prove that the radius of curvature at P equals $2PM$.

11. A particle is sliding down a smooth inverted cycloid (i.e. Fig. 177 upside down); if s is measured from A, the equation of motion is

$$\frac{d^2s}{dt^2}=-g\sin\psi; \quad \text{where } g \text{ is a constant};$$

prove that this can be written

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s.$$

If it starts from rest at $s=4a$, find its velocity in passing through A. Also if the solution is of the form $s=p\cos(nt+\epsilon)$, determine p, n, ϵ and hence find the time taken to reach A.

12. With the data of Ex. 11, if the mass of the particle is m lbs., the pressure on the curve R lbs. at any point P is given by

$$\frac{R-m\cos\psi}{m}=\frac{v^2}{g\rho},$$

where ρ is the radius of curvature at P and v is the speed with which it passes P; express R in terms of ψ.

REVISION PAPERS 18—24

R. 18

1. Differentiate (i) $\cos^2(3x+2)$, (ii) $\sin^{-1}\left(\frac{2x+3}{4}\right)$.

2. For the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, prove that the tangent to the curve makes with OX an angle ϕ such that $\tan\phi=\cot\frac{\theta}{2}$.

3. Use differentials to find $\frac{dy}{dx}$ when (i) $x^2+y^2=a^2$, (ii) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, (iii) $xy=C$.

4. For an alternating current the flow of electricity C is given by $CR+L\frac{dC}{dt}=E\cos pt$, where E, p, R, L are constants; prove that the equation is satisfied by a solution of the form $C=a\cos pt+b\sin pt$ and find a, b.

5. A cup of coffee at temperature 100° C. is placed in a room whose temperature is 15° and it cools 60° in 5 minutes. Find the temperature after another 5 minutes.

R. 19

1. Differentiate (i) $\tan^{-1}\left(\frac{a}{x}\right)$, (ii) $\operatorname{cosec}^{\frac{1}{2}}(1-x)$.

Evaluate $\int_0^a (a^2-x^2)^{\frac{5}{2}}\,dx$.

2. Find the equation of the tangent at the point ϕ to the curve given by $x=a\cos^3\phi$, $y=a\sin^3\phi$. Find also the radius of curvature at this point.

3. A segment of a sphere stands on a base of radius a and its height is h; show that a section of the segment by a plane parallel to the base and at distance y from it is of area $\pi\left[a^2-\frac{y}{h}(a^2-h^2)-y^2\right]$ and deduce that the volume of the segment is $\frac{\pi h}{6}(3a^2+h^2)$.

4. A triangular wall bracket ABC is fastened to a wall with A vertically above B, and C below the level of B; the rods AC, BC stretch so that C descends vertically a small distance δx; express δx in terms of δa, δb; a, b, c being the lengths of BC, CA, AB.

5. Find the position of the centre of gravity of the solid formed by the revolution about OX of $y=5\sin 2x$ between $x=0$ and $x=\frac{\pi}{4}$.

R. 20

1. Evaluate (i) $\frac{d}{dx}\log(1-x)$; (ii) $\frac{d}{dx}\log(\cot 2x)$;

$$\text{(iii)} \int_{-2}^{-1}\frac{dx}{1-2x}; \quad \text{(iv)} \int_0^\infty e^{-x}\,dx.$$

2. Find a differential equation which expresses the fact that the rate at which a liquid passes through a conical paper filter is proportional to the depth of liquid left, the original volume being V_0.

3. The velocity of a body is given by $v=80e^{-\frac{t}{10}}$ in ft./sec. units; find the distance travelled in 5 seconds.

4. Use Simpson's rule to evaluate $\int_5^6 \frac{dx}{x}$ and compare your result with the corresponding logarithm.

5. A rectangular trough 5 feet deep, 3 feet broad is closed by a heavy uniform sluice gate hinged at its upper edge and resting on the bottom of the trough at an angle 60° to the horizontal. If the trough is filled with water (1 cu. ft. weighs 62 lbs.), find the minimum weight of the gate if it is to remain closed.

R. 21

1. Write down the differentials of (i) $x^{-\frac{1}{2}}$; (ii) e^{ax}; (iii) $\log(a-bx)$; (iv) $\tan x$.

2. If $\theta=Ae^{-\lambda t}\sin(at+b)$, where A, a, b are constants, prove that

$$\frac{d^2\theta}{dt^2}+2\lambda\frac{d\theta}{dt}+(\lambda^2+a^2)\theta=0.$$

3. Calculate the area between the curves $y=xe^{-x}$, $y=xe^{x}$ and the line $x=1$.

4. Show that the expression $\tan^{-1}x-\frac{x}{1+x^2}$ is always positive as x changes from 0 to $\frac{\pi}{2}$.

5. A body moves in a straight line so that when its speed is v ft./sec. its retardation is $-kv$ ft./sec.^{2} Initially its speed is v_0 ft./sec.; find how far it travels in t seconds. If $v_0=20$, $k=\frac{1}{4}$, find how long it takes to travel 50 feet, and show that it never reaches a point 80 feet from the starting-point.

R. 22

1. Evaluate (i) $\int (2x^2-5)^2\,dx$, (ii) $\int \frac{6x-5}{3x^2-5x+7}\,dx$,

(iii) $\int (\cos 3\theta+\cos 5\theta)\,d\theta$, (iv) $\int \cos 3\theta \cos 5\theta\,d\theta$.

2. Calculate the amount of £100 in 5 years at 4 per cent. per annum at *continuous* compound interest.

3. A beam supported at its two ends and loaded at the middle sags a distance x given by $x=\frac{kl^3}{bt^3}$, where b, l, t are the breadth, length, thickness respectively of the beam and k is a constant. If the possible errors in b, l, t are each 0·1 per cent., find the approximate maximum percentage error in the deflection x.

4. A body is placed 3 feet from a point O and moves so that its speed is $2x$ ft./sec. when it is x feet from O. Find how far it goes in 2 secs., and the time it takes to go 50 feet.

5. If $P(x_1, y_1)$ is a point on $y=ke^{\frac{x}{a}}$, PN its ordinate, PT the tangent and PG the normal meeting OX at T, G respectively, prove that NT is constant and that $NG=\frac{{y_1}^2}{a}$.

R. 23

1. Evaluate (i) $\int_0^\pi \sin^2 2\theta\,d\theta$, (ii) $\int_0^1 \frac{x^2}{4-x^2}\,dx$.

2. If $e^{(p-q)t}=\frac{\lambda\,(p-x)}{\mu\,(q-x)}$, express $\frac{dx}{dt}$ in terms of x.

3. The motion of a galvanometer needle is given by $\theta=4e^{-\frac{t}{2}}\sin 3t$, where θ is the angle the needle makes with the zero position after t seconds. Find the angular velocity of the needle at any time t and prove that the angles corresponding to the extreme positions of the oscillating needle form a G.P. and find its common ratio.

4. For the curve $ay^2=x^3$ if the length of arc measured from the origin to the point (x, y) is s, prove that

$$\left(\frac{27s}{8a}+1\right)^2=\left(1+\frac{9x}{4a}\right)^3.$$

5. A rope is wound twice round a post and held by a force of 20 lbs. at one end. If the coefficient of friction equals 0·4, find what force is required to make the rope slip.

R. 24

1. If $\dfrac{dx}{dt}=k(a-x)(b-x)$ and if $x=0$ when $t=0$, express t in terms of x.

2. A pane of glass 1 cm. thick absorbs 5 % of the light incident on it. What percentage of light will pass through a sheet of similar glass 1·6 cms. thick?

3. Evaluate (i) $\displaystyle\int\frac{dx}{(x^2-4)(x+3)}$, (ii) $\displaystyle\int_a^b\frac{dx}{\sqrt{\{(x-a)(b-x)\}}}$.

[In (ii) use the substitution $x=a\cos^2\theta+b\sin^2\theta$.]

4. Find the area between the hyperbola $r^2\cos 2\theta=a^2$ and the lines

$$\theta=\pm\frac{\pi}{6}.$$

5. A variable current i flows along a conductor according to the law $i=a\sin(\omega t)$, where t measures the time. Find the mean value of i^2 for the period $t=0$ to $t=\dfrac{2\pi}{\omega}$.

MISCELLANEOUS EXAMPLES 24—31

M. 24

1. The formula $y=R\dfrac{x}{l-x}$ gives the electrical resistance (y) of a coil as measured on a form of Wheatstone Bridge, l being the fixed length of a German silver wire and x the adjustable resistance of a key from one end of the wire. Suppose that with a given value of R an error a is made in x, show that the error in y is approximately $\dfrac{yla}{x(l-x)}$ and hence show that the percentage error in y for a given value of a will be least when the key is in the middle of the wire.

2. Find the condition to be satisfied by the coefficients to secure that

$$x^3+ax^2+bx+c$$

shall increase as x increases for all values of x.

3. The area bounded by the curve $y=f(x)$, the axes of x and y and the ordinate $x=a$ being supposed to rotate about the axis OY, prove that the volume generated is given by the integral $\displaystyle\int_0^a 2\pi xy\,dx$.

The segment of the parabola $y^2=4ax$ bounded by the latus rectum $x=a$ being rotated about the tangent at the vertex, find the volume generated and show as well as you can by sketches the shape of the solid generated. (Army.)

4. In an experiment it was found that the pressure of air on a moving plane was proportional to $\sin\theta\left(1+\dfrac{1}{1+\tan^2\theta}\right)$. Prove that this is equal to $1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$.

Construct the curve $r=1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$ from $\theta=0°$ to $\theta=180°$. Find the maximum and minimum values of r between these limits and the corresponding values of θ. (Army.)

5. Find the area of the curve $r=1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$.

M. 25

1. Sketch the curves $y=e^x$ and $y=xe^x$ between $x=-1$ and $x=1$. Calculate the area enclosed between portions of these two curves and OY.

2. Find the volume of the solid formed by a complete revolution about OX of that portion of $y=a\sin\dfrac{\pi x}{b}$ between $x=0$ and $x=b$.

3. Find (i) $\dfrac{d}{dx}\left(x^{\frac{1}{x}}\right)$, (ii) $\dfrac{d^n}{dx^n}\log_e(x^2+4x+3)$.

4. Find the maxima and minima values of $3\cos^2 x+2(1+\sin x)^2$, distinguishing maxima from minima.

5. For the hyperbola $x^2-y^2=a^2$ prove that the tangent and the radius vector from the centre at any point make complementary angles with OX.

Find the polar equation of this curve.

M. 26

1. A cylinder 25 cms. high and 100 sq. cms. in cross-section has a hole at the bottom. The rate at which water pours out of the hole is proportional to the square root of the depth of water in the cylinder and is 1 cc. per sec. when the cylinder is full. How long will the cylinder take to empty itself?

2. The cross-section of a trough is a parabola with vertex downwards, the latus rectum 4 feet in length lying in the surface. Find the pressure on the end of the trough when it is full of water.

3. The floor of a level quarry is 60 feet below the top. A load of 5 cwt. is drawn slowly across the bottom of the quarry by a wire from the top at A. If the coefficient of friction is 0·2 and x is the distance of the load from the face of the quarry vertically below A, prove that the work done in dragging it from $x=120$ to $x=90$ is $-\int_{120}^{90} \frac{x}{12+x}\,dx$ and that this equals 27 ft. cwts.

4. A tumbler is full of wine. A man sips $\frac{1}{n}$th of the liquid and water is then added until the tumbler is full. If he does this n times where n is infinite, what fraction of the mixture is then wine?

5. The base of a solid is the area between the lines $y=2x$ and $y=-2x$ for the values of x from 0 to 5, and the height at any point whose coordinates are (x, y) is $1+\frac{x}{10}(x-5)$. Find its volume, if the unit on each axis is 1 inch.

M. 27

1. Prove that the volume of a segment of a sphere less than a hemisphere, the radius of whose base is a and whose height is h, is approximately $\frac{1}{2}\pi a^2 h$ and that if $h=\frac{a}{4}$ the error is approximately $2\,{}^0\!/_0$.

2. Find the greatest and least values of $\frac{8}{5+3\sin 2\theta}$.

3. A certain function of x is equal to ax^2 for values of x less than 1 and to $-ax^2+bx-1$ for values of x greater than 1. Find the values of the constants a and b in order that there may be no discontinuity or abrupt changes of slope in the graph of the function at the point $x=1$. With these values of a and b find the values of x for which the function is zero. (Cambridge University.)

4. In a battle between two opposing forces the number of casualties per unit time in either force is proportional to the number of men in the opposing force, the constant factor representing the efficiency of the opposing force which is different for the two sides.

If m and n denote the numbers in the two forces at any time t, a and b the factors representing their efficiency, write down the values of $\frac{dm}{dt}$ and $\frac{dn}{dt}$, and prove that $am^2-bn^2=\text{const.}$

If the two forces contain 10,000 and 5000 men respectively and the smaller force is twice as efficient as the larger, which side will win and how many men will survive if the battle is continued to the bitter end? (Army.)

5. The air pressure on an aeroplane is approximately proportional to $A \sin \theta + B \sin 3\theta$. Find the condition that this expression should have a maximum value for some value of θ between 0° and 90° if $A > 0$.

Find A and B if the expression is a maximum when $\theta = 45°$ and the expression equals 1 when $\theta = 90°$. Find also the maximum value. (Army.)

M. 28

1. Find the rate at which the Napierian logarithm of a number increases compared with the rate at which the number increases when the number is 3. Compare with the tables.

2. The equation of a curve is $s^2 = 4ay$, where s is its length measured from the origin. Find the area of the surface of the solid obtained by the revolution of an arc of length s_1 measured from O of this curve about OX.

3. Show that in the curve $r = a(1 - \cos\theta)$ the angle between the radius vector and the tangent is $\frac{\theta}{2}$.

4. A stone is thrown with a velocity of 80 ft./sec. at an angle of 60° to the horizontal. Taking its highest point as origin and assuming that the path is a parabola, find the equation of the path and the whole length of the path.

5. A quantity of steam expands so as to satisfy the law $pv^{1 \cdot 13} = \text{const.}$ Find the work done in expanding from $v = 3$ to $v = 10$ cu. ft. Given $p = 8640$ lbs./sq. ft. when $v = 1$.

M. 29

1. Show that the curve $s = \sqrt{8ay}$ may also be written in the form $s = 4a \sin \psi$, where ψ is the angle the tangent makes with OX.

2. Find the area of $r = a\theta$ from $\theta = 0$ to $\theta = 2\pi$.

3. A triangular partition in a trough is in the shape of an equilateral triangle of side 2 ft. with its base horizontal. If the level of the water on one side is 1 ft. and on the other side 8 ins. above the vertex, find the magnitude and point of action of the resultant thrust on the partition due to the water.

4. Find the area between the curve $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$, the axis OX and the lines $x=0$, $x=a$.

5. Two large parallel plate condensers at a distance a apart are charged with equal quantities of electricity of opposite sign. The potential V at a point on the positive plate is

$$\int_0^\infty 2\pi\sigma\left(1-\frac{r}{\sqrt{(r^2+a^2)}}\right)dr.$$

Prove that this equals $2\pi a\sigma$, where σ is a constant.

M. 30

1. To measure the distance of an observer O from an accessible point A, a rod 10 ft. long is set up at A at right angles to OA with its mid-point at A and the angle θ subtended by the rod at O is then measured. Find OA in terms of θ and find the error in OA caused by an error $\delta\theta$ in θ. Evaluate when $\theta=72'$ and $\delta\theta=1'$. (Army.)

2. A rhombus consisting of four equal uniform freely-jointed rods, each of weight W, is suspended from an angular point which is connected with the opposite angular point by an elastic string. Show by Virtual Work that in the position of equilibrium the tension in the string equals $2W$.

3. Fig. 178 shows the plan and elevation of a roof. $ABCD$ is part of the curved surface of a cylinder whose radius is a and whose axis is level with the ridge AB and vertically over CD. The other three portions of the roof are also parts of cylinders of the same radius a. Calling the angle AOP, θ, the angle AOR, $\theta+\delta\theta$, and the length CD, b, express the length of PQ and the approximate area of the strip $PQSR$ in terms of a, b, θ and $\delta\theta$. Find by integration the total area of the portion $ABCD$ of the roof.

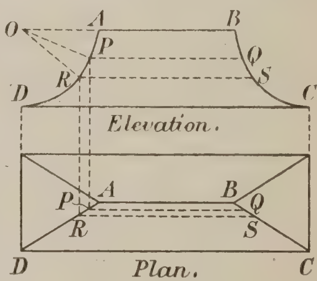

Fig. 178.

4. A water main 4 ft. in diameter is closed by a circular gate. Find the water pressure on it when the main is half full.

5. Find the area of the surface generated by the revolution of a semi-circular arc about a tangent at its mid-point.

M. 31

1. ABC is a triangle of area s; if a and b remain constant but the other parts vary, prove that

$$\frac{\delta s}{s} = -\frac{\delta A \cos C}{\sin A \,.\, \cos B}.$$

2. $ABCD$ is a square. Along AB take any point H. Along BC, CD, DA measure BK, CL, DM respectively each equal to AH. Join AK, BL, CM, DH intersecting one another and forming an inner square $PQRS$. Prove that the ratio of the mean area of $PQRS$ to the area $ABCD$ is

$$\log \frac{e}{2}.$$

3. Find the area between $y = ae^{-bx} \sin x$ and OX from $x=0$ to π.

If $a=1$, $b=0{\cdot}1$, prove that the decrease of area when b is increased by a small quantity λ is approximately $2{\cdot}6\lambda$. (Cambridge University.)

4. Show for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that $\frac{ds}{dx} = \sqrt[3]{\left(\frac{a}{x}\right)}$, and if s is measured from $(0, a)$ find s in terms of x.

5. The magnetic intensity at a point distant r from a long linear conductor carrying a current C is $\int_{-\infty}^{+\infty} \frac{Cr\,dx}{(r^2+x^2)^{\frac{3}{2}}}$; prove this equals $\frac{2C}{r}$.

CHAPTER XVI

COMPLEX NUMBER AND THE HYPERBOLIC FUNCTIONS

THE treatment of certain standard integrals is much simplified by using functions which have a certain analogy with the trigonometric functions: this analogy can best be seen by introducing complex numbers; and because a knowledge of the use of complex numbers is required in all higher branches of mathematics and physics, it is desirable to discuss very shortly their meaning.

Complex Number

If $x^2 = -1$, there is no real value of x which satisfies the equation, because there is no real number whose square is -1. In order therefore to say that every quadratic has a root we introduce what we call *imaginary numbers* and we write

$$x = +\sqrt{-1} \text{ or } -\sqrt{-1}.$$

Our definition of the symbol $\sqrt{-1}$ is the statement

$$\sqrt{-1} \times \sqrt{-1} = -1.$$

By using this symbol, we can now say that every quadratic has two roots.

Thus take the equation $x^2 + a^2 = 0$, we have

$$x^2 = -a^2 = \sqrt{-1} \times \sqrt{-1}\, a^2,$$

$$\therefore\ x = +\sqrt{-1}\, a \text{ or } x = -\sqrt{-1}\, a;$$

or take for example $x^2 + 6x + 13 = 0$, we have

$$x^2 + 6x + 9 = -4 \text{ or } (x+3)^2 = (2\sqrt{-1})^2,$$

$$\therefore\ x + 3 = \pm 2\sqrt{-1} \text{ or } x = -3 \pm 2\sqrt{-1}.$$

Such a number as $-3 + 2\sqrt{-1}$ is called a *complex number* and the number $2\sqrt{-1}$ is called a *pure imaginary*. When we say

that $x=-3+2\sqrt{-1}$ is a root of $x^2+6x+13=0$, we simply mean that if we substitute this expression for x and use the ordinary rules of algebra to simplify and whenever $\sqrt{-1}\times\sqrt{-1}$ occurs we write -1 in its place, then $x=-3+2\sqrt{-1}$ will make the expression $x^2+6x+13$ equal to zero.

Thus if $x=-3+2\sqrt{-1}$, $x^2=(-3+2\sqrt{-1})(-3+2\sqrt{-1})$,

$$\therefore\ x^2=9-6\sqrt{-1}-6\sqrt{-1}+4\sqrt{-1}\times\sqrt{-1}$$
$$=9-12\sqrt{-1}-4=5-12\sqrt{-1},$$
$$\therefore\ x^2+6x+13=5-12\sqrt{-1}+6(-3+2\sqrt{-1})+13$$
$$=5-12\sqrt{-1}-18+12\sqrt{-1}+13=0.$$

Complex numbers are therefore introduced to secure a continuity or generality of statement which would otherwise be unattainable. They constitute a new kind of number, or more accurately a complex number is really a pair of numbers associated together in a particular way. We shall *assume* that the same algebraic operations which can be performed on real numbers can also be performed on complex numbers and shall use this assumption to attach meanings to functions of complex numbers. This is precisely the method adopted in the introduction of fractional indices: we there assumed the general law $x^m\times x^n=x^{m+n}$ and deduced that since $x^{\frac{1}{2}}\times x^{\frac{1}{2}}=x^1=x$ the symbol $x^{\frac{1}{2}}$ means $\sqrt{x}$.

We shall now proceed to attach a meaning to the symbol $e^{\theta\sqrt{-1}}$ on the *assumption* that it obeys the same formal laws of algebra as e^x does when x is real.

In what follows, for the sake of brevity, we shall denote $\sqrt{-1}$ *by* i.

To show that the same meaning* must be attached to the symbol $e^{i\theta}$ as to the expression $\cos\theta+i\sin\theta$

or $\cos\theta+i\sin\theta=e^{i\theta}$.

* The symbol $e^{i\theta}$ really represents a many-valued function, but here it is used to represent the principal value of the function which equals 1 when $\theta=0$.

Let $$y = \cos\theta + i\sin\theta,$$

$$\therefore \frac{dy}{d\theta} = -\sin\theta + i\cos\theta = i^2\sin\theta + i\cos\theta \quad \text{since } i^2 = -1$$

$$= i(i\sin\theta + \cos\theta) = iy,$$

$$\therefore \frac{d\theta}{dy} = \frac{1}{iy} \text{ or } i\frac{d\theta}{dy} = \frac{1}{y},$$

$$\therefore i\theta = \int\frac{dy}{y} = \log y + c.$$

But when $\theta = 0$, $y = \cos 0 + i\sin 0 = 1$;

$$\therefore 0 = \log 1 + c = 0 + c.$$

$$\therefore i\theta = \log y,$$

or $$y = e^{i\theta}.$$

$$\therefore \underline{\cos\theta + i\sin\theta = e^{i\theta}}.$$

Put $\theta = -\phi$ and we have

$$\cos(-\phi) + i\sin(-\phi) = e^{-i\phi},$$

$$\therefore \cos\phi - i\sin\phi = e^{-i\phi}.$$

But the letter used does not make any difference;

$$\therefore \cos\theta - i\sin\theta = e^{-i\theta}.$$

By addition and subtraction we have

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

and by division $$\tan\theta = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}.$$

These relations lead up to the introduction of certain new functions called the *hyperbolic functions*.

Corresponding to $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ we write $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$.

Corresponding to $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ we write $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$.

Corresponding to $\tan\theta = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$ we write $\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$.

For euphony "sinh" is pronounced "shin" and "tanh" is pronounced "than."

In addition we write

$$\frac{1}{\sinh\theta}=\operatorname{cosech}\theta,\quad \frac{1}{\cosh\theta}=\operatorname{sech}\theta,\quad \frac{1}{\tanh\theta}=\coth\theta.$$

"cosech" and "sech" are pronounced "coshec" and "shec" respectively.

The hyperbolic functions have properties very similar to the trigonometric functions.

Example 1.

Prove that (i) $\cosh^2\theta-\sinh^2\theta=1$;

(ii) $\sinh 2\theta=2\sinh\theta\cosh\theta$.

(i) $$\begin{aligned}\cosh^2\theta-\sinh^2\theta&=\tfrac{1}{4}(e^{\theta}+e^{-\theta})^2-\tfrac{1}{4}(e^{\theta}-e^{-\theta})^2\\&=\tfrac{1}{4}(e^{2\theta}+2e^0+e^{-2\theta})-\tfrac{1}{4}(e^{2\theta}-2e^0+e^{-2\theta})\\&=\tfrac{1}{4}\times 4e^0=1.\end{aligned}$$

(ii) $$\begin{aligned}2\sinh\theta\cosh\theta&=2\times\tfrac{1}{2}(e^{\theta}-e^{-\theta})\times\tfrac{1}{2}(e^{\theta}+e^{-\theta})\\&=\tfrac{1}{2}(e^{2\theta}-e^{-2\theta})=\sinh 2\theta.\end{aligned}$$

The relation $\cosh^2\theta-\sinh^2\theta=1$ shows that the point $(a\cosh\theta, b\sinh\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. From this fact the functions derive their names.

The following simple rule (the justification for which is suggested in Example 14, p. 271) enables any hyperbolic function formula to be deduced from the corresponding trigonometrical formula.

In any trigonometrical formula, replace $\sin\theta$, $\cos\theta$, $\tan\theta$ **by** $\sinh\theta$, $\cosh\theta$, $\tanh\theta$ **and wherever a product of sines or an *implied* product of sines occurs, change the sign of the term.**

Thus

$$\cos 2\theta=\cos^2\theta-\sin^2\theta \text{ becomes } \cosh 2\theta=\cosh^2\theta+\sinh^2\theta$$

and

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} \text{ becomes } \tanh(\theta+\phi)=\frac{\tanh\theta+\tanh\phi}{1+\tanh\theta\tanh\phi}$$

for $$\tan\theta\tan\phi\equiv\frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}$$

is an *implied* product of sines.

All such formulae can be proved by simple substitution from the definitions.

EXAMPLES XVI a

1. Simplify:

(i) $(x+iy)(x-iy)$; (ii) $(1+i)^2$; (iii) $\left(\frac{1+i\sqrt{3}}{2}\right)^2$; (iv) $\left(\frac{1-i\sqrt{3}}{2}\right)^3$.

2. Solve the equations:

(i) $x^2+4=0$; (ii) $x^2+8x+25=0$.

3. Show that 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$ are the three roots of $x^3=1$.

4. If a, b are real numbers such that $(a-3)+i(b-5)=0$, find their values.

5. Find the real values of r and θ if $r(\cos\theta+i\sin\theta)=3+4i$.

6. Find the roots of $x^3-x^2+2=0$, given that $x=1+i$ is one root.

7. Prove that

(i) $(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$;

(ii) $(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi)$;

and use (ii) to show that if n is any positive integer

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

8. Use Ex. 7 to show that if q is a positive integer $\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}$ is one of the qth roots of $\cos\theta+i\sin\theta$: show that $\cos\frac{\theta+2\pi}{q}+i\sin\frac{\theta+2\pi}{q}$ is another qth root of $\cos\theta+i\sin\theta$. How many are there?

9. Use Ex. 8 to show that $\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of

$$(\cos\theta+i\sin\theta)^{\frac{p}{q}},$$

where p, q are positive integers.

10. Prove that

$$(\cos\theta+i\sin\theta)^{-1}\equiv\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta=\cos(-\theta)+i\sin(-\theta)$$

and deduce that if n is any positive number

$$(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta.$$

11. Use the relation $\cos\theta+i\sin\theta=e^{i\theta}$

to prove that $\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n$.

12. Use the result of Ex. 11 to prove that

(i) $\cos 3\theta=4\cos^3\theta-3\cos\theta$; (ii) $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

13. Simplify:

$$\text{(i)}\ \frac{\cos 2\theta + i \sin 2\theta}{\cos 3\theta - i \sin 3\theta}; \quad \text{(ii)}\ \frac{1+\cos\theta + i\sin\theta}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}.$$

14. Use the exponential forms for $\cos\theta$ and $\sin\theta$ to prove that

$$\text{(i)}\ \cos(i\theta) = \cosh\theta; \quad \text{(ii)}\ \sin(i\theta) = i\sinh\theta;$$

and deduce the mnemonic given above for obtaining formulae for the hyperbolic functions.

15. Prove that

(i) $\sinh(-x) = -\sinh x$; (ii) $\cosh(-x) = \cosh x$;
(iii) $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$; (iv) $\cosh x \geqslant 1$.

16. What is the value when $x = 0$ of

(i) $\sinh x$, (ii) $\cosh x$, (iii) $\tanh x$?

17. If $y = \sinh^{-1} x$, prove that

(i) $2x = e^y - e^{-y}$; (ii) $e^{2y} - 2x \,.\, e^y - 1 = 0$; (iii) $e^y = x + \sqrt{1+x^2}$.

Hence show that $\sinh^{-1} x = \log(x + \sqrt{1+x^2})$.

18. Use the method of Ex. 17 to prove that

$$\text{(i)}\ \cosh^{-1} x = \pm\log(x + \sqrt{x^2-1}); \quad \text{(ii)}\ \tanh^{-1} x = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

19. Prove that

$$\text{(i)}\ \frac{\cosh 2x - 1}{\cosh 2x + 1} = \tanh^2 x; \quad \text{(ii)}\ \frac{1+\tanh x}{1-\tanh x} = e^{2x};$$

$$\text{(iii)}\ \sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x.$$

20. Prove that

$$(\cosh\theta + \sinh\theta)^n = \cosh(n\theta) + \sinh(n\theta)$$

and simplify $(\cosh\theta - \sinh\theta)^n$.

21. Prove that

$$\text{(i)}\ \operatorname{sech}^2\theta = 1 - \tanh^2\theta; \quad \text{(ii)}\ \operatorname{cosech}^2\theta = \coth^2\theta - 1;$$

$$\text{(iii)}\ \coth\theta + \operatorname{cosech}\theta = \coth\frac{\theta}{2}.$$

22. If $\sinh x = \tan\theta$, express $\cosh x$ and $\tanh x$ in terms of θ.

23. Prove that

(i) $\sec\theta = \cosh[\log(\tan\theta + \sec\theta)]$;
(ii) $\cot\theta = \sinh[\log(\cot\theta + \operatorname{cosec}\theta)]$.

24. Find approximately the value of x for which $\cosh x = 2$.

Differentiation and Integration of $\sinh\theta$ **and** $\cosh\theta$

To prove

$$\frac{d}{d\theta}(\sinh\theta)=\cosh\theta \quad\text{and}\quad \frac{d}{d\theta}(\cosh\theta)=\sinh\theta$$

we have $$\frac{d}{d\theta}(\sinh\theta)=\frac{d}{d\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)=\frac{e^{\theta}+e^{-\theta}}{2}=\cosh\theta$$

and $$\frac{d}{d\theta}(\cosh\theta)=\frac{d}{d\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)=\frac{e^{\theta}-e^{-\theta}}{2}=\sinh\theta.$$

Reversing these results, we have

$$\int\cosh\theta\,.\,d\theta=\sinh\theta \quad\text{and}\quad \int\sinh\theta\,.\,d\theta=\cosh\theta.$$

Integrals of functions involving $\sqrt{a^2+x^2}$ can often be simplified by the substitution $x=a\sinh\theta$ and those involving $\sqrt{x^2-a^2}$ by putting $x=a\cosh\theta$.

Example 2.

Integrate $$\int\frac{dx}{\sqrt{a^2+x^2}}.$$

Put $x=a\sinh\theta$, $\therefore\ dx=a\cosh\theta\,.\,d\theta$

and $$a^2+x^2=a^2+a^2\sinh^2\theta=a^2(1+\sinh^2\theta)=a^2\cosh^2\theta.$$

$$\therefore\ \text{the integral}=\int\frac{a\cosh\theta\,.\,d\theta}{a\cosh\theta}=\int d\theta=\theta.$$

But $$\sinh\theta=\frac{x}{a}.$$

$\therefore$ using the inverse notation

$$\theta=\sinh^{-1}\left(\frac{x}{a}\right).$$

$$\therefore\ \int\frac{dx}{\sqrt{a^2+x^2}}=\sinh^{-1}\left(\frac{x}{a}\right).$$

Example 3.

Find $$\int\sqrt{x^2-a^2}\,dx.$$

Put $x=a\cosh\theta$, $\therefore\ dx=a\sinh\theta\,.\,d\theta$

and $$x^2-a^2=a^2(\cosh^2\theta-1)=a^2\sinh^2\theta.$$

$$\therefore \text{ the integral} = \int a \sinh\theta \,.\, a\sinh\theta \,.\, d\theta$$
$$= a^2 \int \sinh^2\theta \,.\, d\theta$$
$$= \frac{a^2}{2}\int (\cosh 2\theta - 1)\, d\theta$$
$$= \frac{a^2}{2}(\tfrac{1}{2}\sinh 2\theta - \theta)$$
$$= \frac{a^2}{2}(\sinh\theta\cosh\theta - \theta)$$
$$= \frac{a^2}{2}\left[\frac{\sqrt{x^2-a^2}}{a} \,.\, \frac{x}{a} - \cosh^{-1}\left(\frac{x}{a}\right)\right]$$
$$= \tfrac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right).$$

EXAMPLES XVI b

1. Prove that

(i) $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$; (ii) $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$

2. Prove that $\dfrac{d}{dx}\left[\log\left(\tanh\dfrac{x}{2}\right)\right] = \operatorname{cosech} x.$

3. If $y = \sinh^{-1} x$ or $x = \sinh y$, prove that

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$$

and deduce
$$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}.$$

What is
$$\frac{d}{dx}\left[\sinh^{-1}\frac{x}{a}\right]?$$

4. Use the method of Ex. 3 to prove that

(i) $\dfrac{d}{dx}[\cosh^{-1} x] = \pm\dfrac{1}{\sqrt{x^2-1}}$; (ii) $\dfrac{d}{dx}[\tanh^{-1} x] = \dfrac{1}{1-x^2}.$

What is
$$\frac{d}{dx}\left[\tanh^{-1}\frac{x}{a}\right]?$$

5. If
$$y = a\cosh nx + b\sinh nx,$$

prove that
$$\frac{d^2y}{dx^2} - n^2 y = 0.$$

6. What are the values of

$$\int \sinh(nx)\,dx \quad\text{and}\quad \int \cosh(nx)\,dx?$$

Prove that $$\int \tanh(nx)\,dx = \frac{1}{n}\log[\cosh(nx)].$$

7. Prove that $$\int \cosh^2\theta \,.\, d\theta = \frac{\theta}{2} + \tfrac{1}{4}\sinh 2\theta.$$

8. What are the values of

$$\int \operatorname{sech}^2 nx \,.\, dx \quad\text{and}\quad \int \operatorname{cosech}^2 nx \,.\, dx?$$

9. Fig. 179 represents part of the graph of $x^2 - y^2 = 1$. If $ON = \cosh\theta$, prove that $NP = \sinh\theta$ and that the area

$$ANP = \int_0^\theta \sinh^2\theta\, d\theta.$$

Write down the area of triangle ONP and deduce that the area of the sector

$$OAP = \tfrac{1}{2}\theta.$$

What is the corresponding result for the sector of the circle

$$x^2 + y^2 = 1?$$

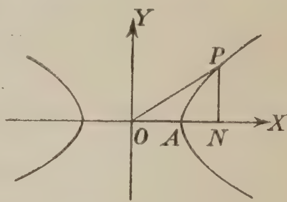

Fig. 179.

10. Prove that

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right).$$

11. Evaluate: (i) $\displaystyle\int \frac{dx}{\sqrt{(9x^2+1)}}$; (ii) $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$.

12. Evaluate:

$$\text{(i)} \int \frac{dx}{\sqrt{[(x+a)^2 - b^2]}}; \quad \text{(ii)} \int \frac{dx}{\sqrt{(x^2+10x+16)}}.$$

13. Evaluate $\displaystyle\int \frac{x^2\,dx}{\sqrt{(1+x^2)}}$ by putting $x = \sinh\theta$.

14. Express the product $\sinh(ax)\sinh(bx)$ as a difference and hence evaluate

$$\int \sinh(ax)\sinh(bx)\,dx.$$

15. If $\cosh u = \sec\theta$, prove that $\dfrac{du}{d\theta} = \sec\theta$ and that

$$u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

16. The equation $y = c \cosh \dfrac{x}{c}$ represents a catenary (see p. 254). If

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

express s in terms of x, given that $s = 0$ when $x = 0$.

17. For the catenary $y = c \cosh \dfrac{x}{c}$, prove with the usual notation:

(i) $y = c \sec \psi$; (ii) $y^2 = c^2 + s^2$; (iii) $x = c \log \tan \left(\dfrac{\pi}{4} + \dfrac{\psi}{2}\right)$.

18. Find the radius of curvature at the point (x, y) on the catenary

$$y = c \cosh \frac{x}{c}.$$

19. Evaluate:

$$\text{(i)} \int x \sinh x \, dx; \quad \text{(ii)} \int e^x \cosh x \, dx.$$

20. Evaluate: (i) $\displaystyle\int \frac{dx}{\sinh x}$; (ii) $\displaystyle\int \sinh^{-1} x \, dx$.

21. Prove that $\dfrac{d}{dx}\left\{\tan^{-1}\left(\tanh \dfrac{x}{2}\right)\right\} = \tfrac{1}{2} \operatorname{sech} x.$

CHAPTER XVII

EXPANSIONS IN SERIES

It is often a matter of practical utility to be able to replace a complicated function of x by one of a simpler type (such as part of a power series in x) which for a special range of values of x approximates in value to the original function.

Example 1.

If x is positive and small compared with 1, $\sqrt[3]{(1+x)} \doteqdot 1+\frac{1}{3}x$ with error less than $\frac{1}{9}x^2$.

We have $$(1+\tfrac{1}{3}x)^3 = 1+x+\tfrac{1}{3}x^2+\tfrac{1}{27}x^3 > 1+x$$

and $$(1+\tfrac{1}{3}x-\tfrac{1}{9}x^2)^3 = 1+x-\frac{x^3(15-x^2)}{81}-\frac{x^6}{729}$$
$$< 1+x \qquad \text{since } x>0.$$

$\therefore$ $\sqrt[3]{(1+x)}$ lies between $1+\frac{1}{3}x$ and $1+\frac{1}{3}x-\frac{1}{9}x^2$.

$\therefore$ $\sqrt[3]{(1+x)} \doteqdot 1+\frac{1}{3}x$ with error less than $\frac{1}{9}x^2$.

This example shows the degree of accuracy attained when the comparatively complicated function $\sqrt[3]{(1+x)}$ is replaced by the first two terms of a power series. By using the Binomial theorem, we can obtain as many terms as we wish in the expansion of $\sqrt[3]{(1+x)}$ as a power series.

If we know that any given function of x **can be expressed** as a power series in x, we can obtain the form of the series by a calculus method.

Example 2.

Given that $$e^x \equiv a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$
where the coefficients a_0, a_1, ... are independent of x, show how to find them.

Since the identity is true for all values of x, we may put $x=0$, then
$$e^0 = a_0, \quad \therefore\ a_0 = 1.$$

Differentiate with respect to x,

$$\therefore\ e^x \equiv a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots;$$

in this identity, put $x=0$,

$$\therefore\ e^0 = a_1 \text{ or } a_1 = 1.$$

Differentiate again,

$$\therefore\ e^x \equiv 2a_2 + 2\,.\,3a_3 x + 3\,.\,4a_4 x^2 + \ldots;$$

put $x=0$, $\quad \therefore\ e^0 = 2a_2$ or $a_2 = \frac{1}{2}$;

repeat the process,

$$\therefore\ e^x \equiv 2\,.\,3a_3 + 2\,.\,3\,.\,4a_4 x + \ldots;$$

put $x=0$, $\quad \therefore\ e^0 = 2\,.\,3a_3$ or $a_3 = \dfrac{1}{2\,.\,3}$,

similarly $a_4 = \dfrac{1}{2\,.\,3\,.\,4}$, $a_5 = \dfrac{1}{2\,.\,3\,.\,4\,.\,5}$, and so on.

$$\therefore\ a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{\underline{|2}},\ a_3 = \frac{1}{\underline{|3}},\ a_4 = \frac{1}{\underline{|4}},\ \ldots,\ a_n = \frac{1}{\underline{|n}},\ \ldots$$

and
$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots.$$

Note. We have made two serious assumptions in establishing the expansion of e^x as a power series of x:

(i) that it is possible to express e^x as an infinite series in x;

(ii) that the differentiation of this power series can be effected term by term.

Both of these assumptions are true in the case of the function e^x, but there are functions for which neither assumption is true (e.g. $\log x$ cannot be expanded as an infinite power series in x) and therefore the proof as given above is incomplete.

We shall now apply the method of this example to any function $f(x)$. The following notation will be used:

$$f'(x),\ f''(x),\ f'''(x),\ \ldots,\ f^n(x),\ \ldots$$

will represent

$$\frac{d}{dx}f(x),\ \frac{d^2}{dx^2}f(x),\ \frac{d^3}{dx^3}f(x),\ \ldots,\ \frac{d^n}{dx^n}f(x),\ \ldots,$$

and $$f'(a),\ f''(a),\ \ldots,\ f^n(a),\ \ldots$$
represent the result of substituting a for x, *after* the differentiation has been completed, in

$$f'(x),\ f''(x),\ \ldots,\ f^n(x),\ \ldots,$$

and in particular $f^n(0)$ represents the result of substituting 0 for x in $f^n(x)$ *after* differentiation.

Maclaurin's Theorem

If $f(x)$ is a function which possesses successive differential coefficients and if it can be expanded as an infinite series in powers of x, then the expansion is

$$f(x) \equiv f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \ldots + \frac{x^n}{\underline{|n}} f^n(0) + \ldots.$$

Let $$f(x) \equiv a_0 + a_1\frac{x}{\underline{|1}} + a_2\frac{x^2}{\underline{|2}} + a_3\frac{x^3}{\underline{|3}} + \ldots + a_n\frac{x^n}{\underline{|n}} + \ldots.$$

[Numerical factors are inserted in the denominators to make the work symmetrical, this does not make the form less general.]

In this identity, put $x = 0$, $\therefore\ f(0) = a_0$.

Differentiate with respect to x,

$$\therefore\ f'(x) = a_1 + a_2\frac{x}{\underline{|1}} + a_3\frac{x^2}{\underline{|2}} + \ldots + a_n\frac{x^{n-1}}{\underline{|n-1}} + \ldots,$$

put $x = 0$, $\qquad \therefore\ f'(0) = a_1$.

Differentiate again,

$$\therefore\ f''(x) = a_2 + a_3\frac{x}{\underline{|1}} + \ldots + a_n\frac{x^{n-2}}{\underline{|n-2}} + \ldots,$$

put $x = 0$, $\qquad \therefore\ f''(0) = a_2$;

clearly we can repeat this process as often as we wish and we obtain

$$a_3 = f'''(0),\ a_4 = f''''(0),\ \ldots,\ a_n = f^n(0),\ \ldots,$$

$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \ldots + \frac{x^n}{\underline{|n}} f^n(0) + \ldots.$$

Note. We have made the same assumptions in this proof as are noted on p. 277.

Example 3.

By assuming Maclaurin's Theorem, prove that

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots.$$

Here

$$\begin{aligned}
f(x) &= \sin x, & \therefore\ f(0) &= \sin 0 = 0;\\
f'(x) &= \cos x, & \therefore\ f'(0) &= \cos 0 = 1;\\
f''(x) &= -\sin x, & \therefore\ f''(0) &= -\sin 0 = 0;\\
f'''(x) &= -\cos x, & \therefore\ f'''(0) &= -\cos 0 = -1;\\
f''''(x) &= \sin x, & \therefore\ f''''(0) &= \sin 0 = 0;
\end{aligned}$$

and this cycle of values now recurs indefinitely,

$$\begin{aligned}
\therefore\ \sin x &= 0 + x + 0 - \frac{x^3}{\underline{|3}} + 0 + \frac{x^5}{\underline{|5}} \dots\\
&= x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \dots.
\end{aligned}$$

EXAMPLES XVII a

By assuming Maclaurin's Theorem, obtain the expansions in Examples 1—9:

1. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \dots.$

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \dots + x^n$, if n is a positive integer.

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ (valid only if $-1 < x \leqslant 1$).

4. $\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \dots.$ $\sinh x = \frac{1}{2}(e^x - e^{-x})$

5. $\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \dots.$

6. $a^x = 1 + x\log a + \frac{x^2(\log a)^2}{\underline{|2}} + \frac{x^3(\log a)^3}{\underline{|3}} + \dots.$

7. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots.$

8. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots.$

9. $\tan^{-1} x = x - \frac{x^3}{3} + \dots$ (valid only if $-1 < x \leqslant 1$).

10. Expand $e^x \cos x$ in powers of x up to x^5.

11. Find the first three terms in the expansion of $\log(1+e^x)$.

12. Expand $e^{\tan x}$ up to x^3.

13. Expand $\log(1+\sin x)$ up to x^4.

14. Show that $\log\left(\dfrac{\sin x}{x}\right) = -\dfrac{x^2}{6} - \dfrac{x^4}{180} \cdots$.

15. Write down the expansion for e^{ix}: then equate real and imaginary parts in the relation $\cos x + i \sin x \equiv e^{ix}$.

16. [Taylor's Theorem.] Assuming that $f(x+h)$ can be expanded as a power series in h, prove by the method used above for Maclaurin's Theorem that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \ldots + \frac{h^n}{\underline{|n}} f^n(x) + \ldots.$$

Other methods of expansion or summation

When a function of x is expressed as an infinite series in powers of x, it is true *under certain conditions* to say that the differential coefficient of the function equals the result of differentiating the series term by term and the integral of the function is equal to the result of integrating the series term by term. It is beyond the scope of this book to investigate what these conditions are, we shall however illustrate the process by two examples.

Example 4.

Obtain a series for $\sin^{-1}(x)$ if $-1 < x < 1$.

By the binomial theorem, since $-1 < x < 1$,

$$\frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1\,.\,2}x^4 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\,.\,2\,.\,3}x^6 + \ldots$$

$$= 1 + \tfrac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \ldots.$$

Integrate with respect to x,

$$\therefore \int \frac{dx}{\sqrt{(1-x)^2}} = \sin^{-1} x = c + x + \tfrac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \ldots.$$

If when $x=0$ we take the principal value of $\sin^{-1} x$ as 0, we have $c=0$.

$$\therefore \sin^{-1} x = x + \tfrac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \ldots.$$

Example 5.

If $-1<x<1$, sum the series

$$1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}+\ldots.$$

Since $-1<x<1$, we have

$$\frac{1}{(1-x)^2}=(1-x)^{-2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots,$$

$$\therefore\ \frac{x}{(1-x)^2}=x+2x^2+3x^3+\ldots+nx^n+\ldots;$$

differentiate with respect to x,

$$\therefore\ \frac{1}{(1-x)^2}+\frac{2x}{(1-x)^3}=1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}+\ldots,$$

$$\therefore\ \text{required series}=\frac{1-x+2x}{(1-x)^3}=\frac{1+x}{(1-x)^3}.$$

EXAMPLES XVII b

1. Obtain a series for $\tan^{-1}x$ by using

$$\tan^{-1}x=\int_0^x\frac{dx}{1+x^2}$$

when $-1<x<1$.

2. Obtain a series for $\log(1+x)$ if $-1<x<1$ from

$$\log(1+x)=\int_0^x\frac{dx}{1+x}.$$

3. What relation can be obtained by integrating $\dfrac{2}{1-x^2}$?

4. If $-1<x<1$, sum the series

$$1.2+2.3x+3.4x^2+4.5x^3+\ldots.$$

5. If $-1<x<1$, sum the series

$$\frac{x^2}{1.2}-\frac{x^3}{2.3}+\frac{x^4}{3.4}-\ldots.$$

6. If $-1<x<1$, sum the series

$$1+\frac{1.3}{2}x^2+\frac{1.3.5}{2.4}x^4+\frac{1.3.5.7}{2.4.6}x^6+\ldots.$$

7. Sum the series

$$1-3\frac{x^2}{\underline{|2}}+5\frac{x^4}{\underline{|4}}-7\frac{x^6}{\underline{|6}}+\ldots.$$

8. Sum the series

$$1+\frac{2x^2}{\underline{|3}}+\frac{3x^4}{\underline{|5}}+\frac{4x^6}{\underline{|7}}+\frac{5x^8}{\underline{|9}}+\dots.$$

9. From the series

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots,$$

where $-1<x<1$, deduce the sum of the series

$$x-\frac{2x^3}{3}+\frac{3x^5}{5}-\frac{4x^7}{7}+\dots.$$

10. If $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+x=e^{-2t}$ and if when $t=0$, $x=0$ and $\dfrac{dx}{dt}=0$, show that

$$x=\tfrac{1}{2}t^2-\tfrac{2}{3}t^3+\tfrac{11}{24}t^4-\dots.$$

11. Expand $\sin^2 x$ in powers of x. [Use multiple angles.]

12. Expand $\sin x\cos x$ in powers of x.

13. [Huygen's rule.] If a is the chord of the arc AB of a circle and if b is the chord of half the arc AB, then the length of the arc is approximately $\frac{1}{3}(8b-a)$. If the arc AB subtends θ radians at the centre of the circle, show that the error per cent. is approximately $\dfrac{\theta^4}{75}$.

14. Expand $\tanh^{-1}x$ as a power series in x if $-1<x<1$.

15. Expand $\log[x+\sqrt{(1+x^2)}]$ as a power series in x if $-1<x<1$.

In the work of this chapter we have so far given no proof of the *existence* of the various expansions considered. We have merely shown what forms they must take, *if they exist at all.*

Now when we say that a function $f(x)$ is equal to an infinite series in powers of x,

$$a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots,$$

what we mean is that the difference between the value of $f(x)$ and the value of the n terms

$$a_0+a_1x+\dots+a_{n-1}x^{n-1}$$

can be made as small as we please by taking a sufficiently large value of n for any value of x in the range of values of x considered.

If therefore we can obtain an expression for this difference, we may then be able to see if it satisfies this essential test: if it does not do so, the expansion is untrue.

Example.

Under what conditions does

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots\to\infty\,?$$

Now by actual division or by taking the sum of a G. P.

$$1+x+x^2+\ldots+x^{n-1}=\frac{1-x^n}{1-x},$$

$$\therefore \text{ the difference } \frac{1}{1-x}-\{1+x+x^2+\ldots+x^{n-1}\}=\frac{1}{1-x}-\frac{1-x^n}{1-x}=\frac{x^n}{1-x}.$$

The test therefore is this: can we find a value of n such that $\dfrac{x^n}{1-x}$ is less than any given amount however small, for this and all greater values of n? If we can do so, we may regard

$$1+x+x^2+\ldots+x^{n-1}$$

as an approximation for $\dfrac{1}{1-x}$ and the approximation can be made as close as we please by taking n large enough.

Now obviously we cannot always do this; for example if $x=2$,

$$\frac{x^n}{1-x}=\frac{2^n}{1-2}=-2^n,$$

and so the larger n becomes the greater is the error in taking

$$1+x+x^2+\ldots+x^{n-1}$$

to represent $\dfrac{1}{1-x}$.

But if x is any fraction *between* 1 and -1, for example $\frac{9}{10}$, we can make

$$\frac{x^n}{1-x}=10\,(0{\cdot}9)^n$$

as small as we please by taking n large enough.

We therefore say that if x is any fraction between 1 and -1, the function $\dfrac{1}{1-x}$ and the infinite series $1+x+x^2+\ldots$ are equal: they are not equal if $x\geqq 1$ or if $x\leqq -1$.

Before trying to apply this test to Maclaurin's Theorem, we must establish the following result.

If when x varies from a to $a+h$ where $h>0$, the greatest and least values of the continuous function $\phi(x)$ are M and m, then

$$Mh > \int_a^{a+h} \phi(x)\,dx > mh.$$

Let A, B be the points on the graph of $y=\phi(x)$ whose abscissae are a, $a+h$, and let GN, gn represent the greatest and least values of $\phi(x)$ between A and B.

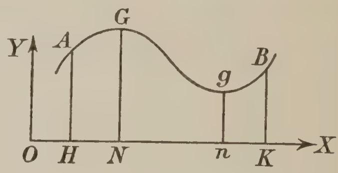

Fig. 180.

Then

$$HK = OK - OH = a+h-a = h,$$

$$GN = M, \quad gn = m.$$

$$\int_a^{a+h} \phi(x)\,dx = \text{the curvilinear area } HAGgBK.$$

This is obviously less than $GN \times HK$ or Mh and obviously greater than $gn \times HK$ or mh,

$$\therefore\; Mh > \int_a^{a+h} \phi(x)\,dx > mh.$$

We shall now use the method of integrating by parts to obtain the expansion of a function.

Example 6.

Expand e^a in powers of a, where $a>0$.

We have $e^a - 1 = \int_0^a e^x\,dx = -\int_{x=0}^a e^x\,d(a-x)$

$$= -\Big[e^x(a-x)\Big]_0^a + \int_0^a (a-x)\,d(e^x)$$

$$= -(0-a) + \int_0^a (a-x)\,e^x\,dx$$

$$= a - \int_0^a e^x\,d\left(\frac{(a-x)^2}{2}\right)$$

$$= a - \left[e^x \cdot \frac{(a-x)^2}{2}\right]_0^a + \int_0^a \frac{(a-x)^2}{2}\,d(e^x)$$

$$= a - \left(0 - \frac{a^2}{2}\right) + \int_0^a \frac{(a-x)^2}{2}\,e^x\,dx$$

$$= a + \frac{a^2}{2} - \int_0^a e^x\,d\left(\frac{(a-x)^3}{2.3}\right)$$

$$=a+\frac{a^2}{2}-\left[e^x\cdot\frac{(a-x)^3}{2\,.\,3}\right]_0^a+\int_0^a\frac{(a-x)^3}{2\,.\,3}\,d\,(e^x)$$
$$=a+\frac{a^2}{2}+\frac{a^3}{2\,.\,3}+\int_0^a\frac{(a-x)^3}{2\,.\,3}\,e^x\,dx.$$

Clearly we can repeat this process as often as we like, we thus obtain

$$e^a=1+a+\frac{a^2}{\lfloor 2}+\frac{a^3}{\lfloor 3}+\frac{a^4}{\lfloor 4}+\ldots+\frac{a^n}{\lfloor n}+\int_0^a\frac{(a-x)^n}{\lfloor n}\,e^x\,dx.$$

We cannot evaluate the integral obtained here but we can find values between which it must lie: for when x varies from 0 to a, $(a-x)^n$ lies between a^n and 0, also e^x lies between e^0 and e^a or 1 and e^a.

$\therefore\ \dfrac{(a-x)^n}{\lfloor n}\,e^x$ lies between 0 and $\dfrac{a^n}{\lfloor n}\,e^a$.

$\therefore\ \displaystyle\int_0^a\frac{(a-x)^n}{\lfloor n}\,e^x\,dx>0$ and $<\dfrac{a^n e^a}{\lfloor n}\times a$ or $\dfrac{a^{n+1}e^a}{\lfloor n}$.

$$\therefore\ e^a>1+a+\frac{a^2}{\lfloor 2}+\ldots+\frac{a^n}{\lfloor n}$$

and
$$e^a<1+a+\frac{a^2}{\lfloor 2}+\ldots+\frac{a^n}{\lfloor n}+\frac{a^{n+1}e^a}{\lfloor n}.$$

$\therefore$ the error in replacing e^a by $1+a+\dfrac{a^2}{\lfloor 2}+\ldots+\dfrac{a^n}{\lfloor n}$ is less than $\dfrac{a^{n+1}e^a}{\lfloor n}$.

We need $\therefore$ only consider whether this error can be made as small as we please by taking a large enough value for n.

Now if we take $(n+2)$ terms instead of $(n+1)$ terms the error is $\dfrac{a^{n+2}e^a}{\lfloor n+1}$, i.e. it is $\dfrac{a}{n+1}\times$ the former error; and so as soon as $n>2a$, the addition of each successive term makes the new error less than half the old error and $\therefore$ we can make the error as small as we please by carrying out the halving process a sufficient number of times.

We therefore say that for *all* positive values of a

$$e^a=1+a+\frac{a^2}{\lfloor 2}+\frac{a^3}{\lfloor 3}+\ldots+\frac{a^n}{\lfloor n}+\ldots\rightarrow\infty.$$

A similar proof may be used if $a<0$; in that case the integral may be negative, but the error remains less than the positive value of $\dfrac{a^{n+1}e^a}{\lfloor n}$.

We shall use the method of integrating by parts followed in this example to give a more accurate statement of Maclaurin's Theorem.

Maclaurin's Theorem

If as x varies from 0 to a, $f(x)$ is continuous and possesses continuous successive differential coefficients, then

$$f(a) = f(0) + a f'(0) + \frac{a^2}{\underline{|2}} f''(0) + \dots$$
$$+ \frac{a^n}{\underline{|n}} f^n(0) + \int_0^a \frac{(a-x)^n}{\underline{|n}} f^{n+1}(x)\,dx.$$

We have

$$\begin{aligned}
&f(a) - f(0)\\
&= \int_0^a f'(x)\,dx = -\int_{x=0}^a f'(x)\,d(a-x)\\
&= -\Big[f'(x)(a-x)\Big]_0^a + \int_0^a (a-x)\,d\,[f'(x)]\\
&= -[0 - a f'(0)] + \int_0^a (a-x) f''(x)\,dx\\
&= a f'(0) - \int_0^a f''(x)\,d\left(\frac{(a-x)^2}{2}\right)\\
&= a f'(0) - \left[f''(x)\,.\,\frac{(a-x)^2}{2}\right]_0^a + \int_0^a \frac{(a-x)^2}{2}\,d\,[f''(x)]\\
&= a f'(0) - \left[0 - \frac{a^2}{2} f''(0)\right] + \int_0^a \frac{(a-x)^2}{2} f'''(x)\,dx\\
&= a f'(0) + \frac{a^2}{2} f''(0) - \int_0^a f'''(x)\,d\left[\frac{(a-x)^3}{2\,.\,3}\right]\\
&= a f'(0) + \frac{a^2}{2} f''(0) - \left[f'''(x)\,.\,\frac{(a-x)^3}{2\,.\,3}\right]_0^a + \int_0^a \frac{(a-x)^3}{2\,.\,3}\,d\,[f'''(x)]\\
&= a f'(0) + \frac{a^2}{2} f''(0) + \frac{a^3}{2\,.\,3} f'''(0) + \int_0^a \frac{(a-x)^3}{2\,.\,3} f''''(x)\,dx.
\end{aligned}$$

Clearly we can repeat this process as often as we like, we then obtain

$$f(a) = f(0) + a f'(0) + \frac{a^2}{\underline{|2}} f''(0) + \frac{a^3}{\underline{|3}} f'''(0) + \dots$$
$$+ \frac{a^n}{\underline{|n}} f^n(0) + \int_0^a \frac{(a-x)^n}{\underline{|n}} f^{n+1}(x)\,dx.$$

Whether this expansion is of any use depends on whether $\int_0^a \frac{(a-x)^n}{\lfloor n} f^{n+1}(x)\,dx$ decreases indefinitely as n increases.

Now when x varies from 0 to a, $(a-x)^n$ varies from a^n to 0; suppose that $f^{n+1}(x)$ lies between the numbers $+M$ and $-M$, then $\frac{(a-x)^n}{\lfloor n} f^{n+1}(x)$ lies between $\frac{Ma^n}{\lfloor n}$ and $-\frac{Ma^n}{\lfloor n}$.

$\therefore$ the integral lies between $\pm \frac{Ma^{n+1}}{\lfloor n}$.

Hence we can say that if we replace $f(a)$ by the terminating series

$$f(0) + af'(0) + \frac{a^2}{\lfloor 2} f''(0) + \ldots + \frac{a^n}{\lfloor n} f^n(0),$$

the error is less than $\frac{Ma^{n+1}}{\lfloor n}$, where M is the greatest value of $f^{n+1}(x)$ when x varies from 0 to a.

If this error can be made as small as we please by taking n sufficiently large, we say that

$$f(a) = f(0) + af'(0) + \frac{a^2}{\lfloor 2} f''(0) + \ldots$$
$$+ \frac{a^n}{\lfloor n} f^n(0) + \frac{a^{n+1}}{\lfloor n+1} f^{n+1}(0) + \ldots \to \infty.$$

Example 7.

Expand $(z+a)^n$ where n is a positive integer.

Here $f(a) \equiv (z+a)^n$, $\quad \therefore f(0) = z^n$;

$f'(a) = n(z+a)^{n-1}$, $\quad \therefore f'(0) = nz^{n-1}$;

$f''(a) = n(n-1)(z+a)^{n-2}$, $\quad \therefore f''(0) = n(n-1)z^{n-2}$;

and so on $\quad$ ....

$f^n(a) = \lfloor n$, $\quad \therefore f^n(0) = \lfloor n$;

$f^{n+1}(a) = 0$, $\quad \therefore \int_0^a \frac{(a-x)^n}{\lfloor n} f^{n+1}(x)\,dx = 0.$

$$\therefore (z+a)^n = z^n + a\,.\,nz^{n-1} + \frac{a^2}{\lfloor 2}\,.\,n(n-1)z^{n-2} + \ldots\ldots + \frac{a^n}{\lfloor n}\,.\,\lfloor n + 0$$
$$= z^n + nz^{n-1}a + \frac{n(n-1)}{\lfloor 2} z^{n-2}a^2 + \ldots\ldots + a^n.$$

Maclaurin's Theorem can be written in a different form, known as Taylor's Theorem.

Taylor's Theorem

If as z varies from x to $x+h$, $f(z)$ possesses successive differential coefficients, then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\dots$$

$$+\frac{h^n}{\lfloor n}f^n(x)+\int_x^{x+h}\frac{(x+h-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

Let $f(x+h)\equiv\phi(h)$ so that $f(x)=\phi(0)$.

Then

$$\phi'(h)=\frac{d}{dh}f(x+h)=\frac{df(x+h)}{d(x+h)}\times\frac{d(x+h)}{dx}=f'(x+h),$$

$$\therefore\ \phi'(0)=f'(x).$$

Similarly $\phi''(h)=f''(x+h)$, $\phi''(0)=f''(x)$, etc.

By Maclaurin's Theorem

$$\phi(h)=\phi(0)+h\phi'(0)+\frac{h^2}{\lfloor 2}\phi''(0)+\dots$$

$$+\frac{h^n}{\lfloor n}\phi^n(0)+\int_0^h\frac{(h-u)^n}{\lfloor n}\phi^{n+1}(u)\,du.$$

Now

$$\int_0^h\frac{(h-u)^n}{\lfloor n}\phi^{n+1}(u)\,du=\int_0^h\frac{(h-u)^n}{\lfloor n}f^{n+1}(x+u)\,du \qquad \text{put } x+u=z$$

$$=\int_x^{x+h}\frac{(h+x-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

$$\therefore f(x+h)=\phi(h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\dots$$

$$+\frac{h^n}{\lfloor n}f^n(x)+\int_x^{x+h}\frac{(h+x-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

EXAMPLES XVII c

1. Show that $\sin a = a - \frac{a^3}{3!} + \int_0^a \frac{(a-x)^3}{3!} \sin x \, dx,$

and deduce that the error in replacing $\sin a$ by $a - \frac{a^3}{3!}$ is less than $\frac{a^4}{6}$.

2. Show that $\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \int_0^a \frac{(a-x)^3}{(1+x)^4} dx,$

and deduce that the error in replacing $\log(1+a)$ by $a - \frac{a^2}{2} + \frac{a^3}{3}$ is less than a^4, if $0 < a < 1$.

3. Show that the error in replacing $\cos x$ by $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$ is less than $\frac{x^{2n+1}}{(2n+1)!}$.

4. If $f(x)$ and $f'(x)$ are continuous as x varies from a to $a+h$, use the relation $f(a+h) - f(a) = \int_a^{a+h} f'(x)\, dx$ to prove that there exists a number θ between 0 and 1 such that $f(a+h) - f(a) = hf'(a+\theta h)$.

5. By drawing the graph of $y = f(x)$, interpret geometrically the meaning of θ in the relation $\frac{f(a+h) - f(a)}{h} = f'(a+\theta h)$.

Indeterminate Forms

Example 8.

Find the limit of $\frac{e^x - e^{-x}}{x}$ when $x \to 0$.

If we put $x=0$ in $\frac{e^x - e^{-x}}{x}$ we obtain $\frac{1-1}{0}$ or $\frac{0}{0}$ which is meaningless: the function has therefore no meaning when $x=0$, but it has a definite value for every value of x other than $x=0$, and as $x \to 0$ the function may tend to a definite value.

Now $$e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)$$
$$= 2\left(x + \frac{x^3}{3!} + \ldots\right).$$

$$\therefore \frac{e^x - e^{-x}}{x} = 2\left(1 + \frac{x^2}{3!} + \ldots\right) \text{ provided } x \neq 0.$$

$$\therefore \text{ as } x \to 0, \frac{e^x - e^{-x}}{x} \to 2 \text{ and we say } \operatorname*{Lt}_{x\to 0} \frac{e^x - e^{-x}}{x} = 2.$$

We shall now prove the following theorem :

If $f(x)$ and $\phi(x)$ are two functions which possess successive differential coefficients and such that when $x=a$, $f(a)=0=\phi(a)$, then

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{\phi'(x)}.$$

By hypothesis, $\dfrac{f(x)}{\phi(x)}$ has no meaning when $x=a$ but it may tend to a definite limit when $x\to a$.

By Taylor's Theorem

$$\frac{f(a+h)}{\phi(a+h)}=\frac{f(a)+hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\dots}{\phi(a)+h\phi'(a)+\frac{h^2}{\underline{|2}}\phi''(a)+\dots}$$

$$=\frac{hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\dots}{h\phi'(a)+\frac{h^2}{\underline{|2}}\phi''(a)+\dots}\qquad\text{since } f(a)=0=\phi(a)$$

$$=\frac{f'(a)+\frac{h}{\underline{|2}}f''(a)+\dots}{\phi'(a)+\frac{h}{\underline{|2}}\phi''(a)+\dots}\quad\text{if } h\neq 0.$$

$\therefore$ when $h\to 0$, $\qquad \dfrac{f(a+h)}{\phi(a+h)}\to\dfrac{f'(a)}{\phi'(a)}$.

$$\therefore \operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)}{\phi(a+h)}=\frac{f'(a)}{\phi'(a)}.$$

If however we also have

$$f'(a)=0=\phi'(a),$$

we obtain in the same way

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\frac{f''(a)}{\phi''(a)}$$

and so on.

Example 9.

Evaluate $$\operatorname*{Lt}_{x \to \infty} x \sin\left(\frac{1}{x}\right),$$

put $x = \frac{1}{u}$; when $x \to \infty$, $u \to 0$,

$$\therefore \operatorname*{Lt}_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \operatorname*{Lt}_{u \to 0} \frac{1}{u} \sin u = \operatorname*{Lt}_{u \to 0} \frac{\sin u}{u} = 1.$$

EXAMPLES XVII d

Evaluate the following limits:

1. $\operatorname*{Lt}_{x \to 1} \dfrac{x^5 - x^4}{x^3 + 3x^2 - 4}$. **2.** $\operatorname*{Lt}_{x \to 1} \dfrac{\log x}{x - 1}$. **3.** $\operatorname*{Lt}_{x \to 0} \dfrac{e^x - e^{-x}}{x}$.

4. $\operatorname*{Lt}_{x \to 0} \dfrac{1 - \cos 2x}{1 - \cos 3x}$. **5.** $\operatorname*{Lt}_{x \to \frac{\pi}{2}} \dfrac{\frac{\pi}{2} - x}{\cos 3x}$. **6.** $\operatorname*{Lt}_{x \to 0} \sin x \log x$.

7. $\operatorname*{Lt}_{x \to \infty} x^2 \left[1 - \cos\left(\frac{1}{x}\right)\right]$. **8.** $\operatorname*{Lt}_{x \to \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 + 2x})$.

9. $\operatorname*{Lt}_{x \to \infty} \dfrac{x + \log x}{x \log x}$.

10. If $\cos 2n\theta \equiv a_0 + a_1 \sin^2\theta + a_2 \sin^4\theta + \ldots + a_n \sin^{2n}\theta$, where n is an integer, evaluate a_0 by putting $\theta = 0$ and evaluate a_1 by taking the limit when $\theta \to 0$ of the relation

$$\frac{\cos 2n\theta - a_0}{\sin^2\theta} \equiv a_1 + a_2 \sin^2\theta + \ldots.$$

11. If $\dfrac{\sin 2n\theta}{\sin\theta\cos\theta} \equiv a_0 + a_1 \sin^2\theta + a_2 \sin^4\theta + \ldots + a_{n-1} \sin^{2n-2}\theta$, where n is an integer, evaluate a_0 and a_1 by the limit method indicated in Ex. 10.

MISCELLANEOUS EXAMPLES 32—35

M. 32

1. (i) If $p = c \,.\, e^{\frac{\theta}{\theta - a}}$ where c, a are constants, find $\dfrac{1}{p}\dfrac{dp}{d\theta}$ in terms of θ.

(ii) Expand $\log(\cosh x)$ in powers of x as far as x^4.

2. A particle of mass 1 lb. falls vertically under gravity and the air resistance is kv^2 lbs. when the velocity is v ft. sec.; using the equation $\dfrac{dv}{dt} = g(1 - kv^2)$, prove that in t secs. it falls $\dfrac{1}{gk} \log[\cosh(gt\sqrt{k})]$ feet.

3. Find the length of the subtangent to the curve $y=\dfrac{x^2}{2a-x}$ at the point (a, a).

4. A shaft is rotating under a variable couple $a\sin^2\theta$ where θ is the angle turned through at any time. Find the amount of work done per revolution.

5. If $x=f(t)$ and $y=\phi(t)$ and if $\dfrac{d^2y}{dx^2}=0$, prove that

$$\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}=\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}.$$

If a curve is given by $x=a\cos t+\frac{1}{2}b\cos 2t$, $y=a\sin t+\frac{1}{2}b\sin 2t$, prove that the points for which $\dfrac{d^2y}{dx^2}=0$ are given by $\cos t=-\dfrac{a^2+2b^2}{3ab}$. What geometrical condition is satisfied at these points?

M. 33

1. If $y^{\frac{1}{m}}=x+\sqrt{(1+x^2)}$, prove that $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=m^2y$.

2. (i) If $0<a<1$ show that $\displaystyle\int_0^a x\log(1+x)\,dx<\frac{a^3}{3}$.

(ii) Prove that $\displaystyle\int_0^1 x\log\left(1+\frac{x}{2}\right)dx=\tfrac{3}{4}-\tfrac{3}{2}\log\left(\tfrac{3}{2}\right)$.

3. If the tangent at a point P of a curve cuts the x-axis OX at T and if for all positions of P the ordinate of P equals OT, find the equation of the curve.

4. A body has a circular base centre O, radius a, and the height at any point P of the base is $a\cos\dfrac{\pi x}{2a}$ where $OP=x$. Calculate the volume of the body and the height of its centre of gravity above the base.

5. A circle of radius a rolls without slipping on the outside of a fixed circle of radius $2a$, centre O; a point P on the rim of the moving circle is initially in contact with the fixed circle at B, OA is a radius perpendicular to OB; when the moving circle touches the fixed circle at Q, show that the tangent to the locus of P is perpendicular to QP and if p is the length of the perpendicular from O to this tangent and if the perpendicular makes with OA an angle ψ, prove that the locus of P is given by $p=4a\sin\dfrac{\psi}{2}$ or $p=4a\cos\dfrac{\psi}{2}$. Use the relation $r\dfrac{dr}{dp}=\rho=p+\dfrac{d^2p}{d\psi^2}$ to prove that the pedal equation of the locus of P is $4(r^2-4a^2)=3p^2$.

M. 34

1. A regular pyramid, vertex O, stands on a square base $ABCD$. Its net is formed by cutting down the edges OA, OB, OC, OD and folding the faces flat in the plane of the base. If the net had been obtained from a sheet of cardboard 2 feet square with its sides parallel to AB, BC, what length should AB have to give the maximum volume for the pyramid?

2. A cylinder 2″ in diameter (see Fig. 181) is provided with a groove whose section is a semicircle diameter $\frac{1}{2}$″; find the area of the surface of the groove.

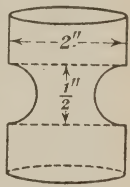

Fig. 181.

3. Find a point on the curve $2y=5+x^4$ such that the tangent at that point passes through the origin.

4. By using the formula $a^2=b^2+c^2-2bc\cos A$ for the triangle ABC, prove that if A increases by $1'$ and if b, c remain constant, the increase in a is approximately $\dfrac{\pi p}{10800}$, where p is the perpendicular from A to BC.

5. Show that $y=x\cos 2x+2x^2\sin 2x$ is a solution of the equation $\dfrac{d^2y}{dx^2}+4y=16x\cos 2x$ and that $y=a\sin 2x+b\cos 2x+x\cos 2x+2x^2\sin 2x$ is the general solution.

M. 35

1. ACB is a horizontal line and BE a vertical line: a rod CD is hinged at C and turns about C in a vertical plane at the uniform rate of 1 revolution per minute. BP is the shadow of CD cast by a light at A on the wall BE; $AC=10$ feet, $AB=24$ feet, $CD=5$ feet. Find the velocity of P when D is 3 feet above AC.

2. A sphere of radius a has a hollow concentric spherical cavity of radius $\dfrac{2a}{3}$. Show that a plane whose distance from the centre is $\dfrac{19a}{45}$ divides the sphere into two portions whose volumes are in the ratio 3 : 1.

3. Through the fixed point (a, b) a line is drawn so that the portion intercepted between the axes is a minimum; prove that its length is

$$(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}.$$

4. If $y=4x^2(1-x^2)$, find the square root of the mean value of y^2 for equal intervals of x between $x=0$ and $x=1$. Interpret the result geometrically, using the idea of a volume.

5. Find the ordinate of the point on the curve $y=a\sin\dfrac{x}{p}$ at which the centre of the circle of curvature lies on the x-axis.

LOGARITHMIC TABLES

NAPIERIAN LOGARITHMS (BASE e)

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	0·0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	10	19	29	38	48	57	67	76	86
1·1	0·0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	0·1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	0·2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	44	52	59	67
1·4	0·3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	21	28	35	41	48	55	62
1·5	0·4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	7	13	19	26	32	39	45	52	58
1·6	0·4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	55
1·7	0·5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	23	29	34	40	46	52
1·8	0·5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	22	27	32	38	43	49
1·9	0·6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	0·6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	20	24	29	34	39	44
2·1	0·7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	5	9	14	19	23	28	33	37	42
2·2	0·7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	4	9	13	18	22	27	31	36	40
2·3	0·8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	4	9	13	17	21	26	30	34	38
2·4	0·8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	4	8	12	16	20	24	29	33	37
2·5	0·9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	4	8	12	16	20	24	27	31	35
2·6	0·9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	4	8	11	15	19	23	26	30	34
2·7	0·9933	9969	$\overline{0006}$	$\overline{0043}$	$\overline{0080}$	$\overline{0116}$	$\overline{0152}$	$\overline{0188}$	$\overline{0225}$	$\overline{0260}$	4	7	11	15	18	22	26	29	33
2·8	1·0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	4	7	11	14	18	21	25	28	32
2·9	1·0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	3	7	10	14	17	20	24	27	31
3·0	1·0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	3	7	10	13	16	20	23	26	30
3·1	1·1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	3	6	10	13	16	19	22	25	29
3·2	1·1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	3	6	9	12	15	18	21	25	28
3·3	1·1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	3	6	9	12	15	18	21	24	27
3·4	1·2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	3	6	9	12	14	17	20	23	26
3·5	1·2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	3	6	8	11	14	17	20	22	25
3·6	1·2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	3	5	8	11	14	16	19	22	25
3·7	1·3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	3	5	8	11	13	16	19	21	24
3·8	1·3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	3	5	8	10	13	16	18	21	23
3·9	1·3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	3	5	8	10	13	15	18	20	23
4·0	1·3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	2	5	7	10	12	15	17	20	22
4·1	1·4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	2	5	7	10	12	14	17	19	22
4·2	1·4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	2	5	7	9	12	14	16	19	21
4·3	1·4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	2	5	7	9	12	14	16	18	21
4·4	1·4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	2	4	7	9	11	13	16	18	20
4·5	1·5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	2	4	7	9	11	13	15	18	20
4·6	1·5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	2	4	6	9	11	13	15	17	19
4·7	1·5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	2	4	6	8	11	13	15	17	19
4·8	1·5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	2	4	6	8	10	12	14	16	19
4·9	1·5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	2	4	6	8	10	12	14	16	18
5·0	1·6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	2	4	6	8	10	12	14	16	18
5·1	1·6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	2	4	6	8	10	12	14	16	18
5·2	1·6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	2	4	6	8	10	12	13	15	17
5·3	1·6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	2	4	6	8	9	11	13	15	17
5·4	1·6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	2	4	6	7	9	11	13	15	17

NAPIERIAN LOGARITHMS OF 10^n

n	1	2	3	4	5	6	7	8	9	10
$\log_e 10^n$	2·3026	4·6052	6·9078	9·2103	11·5129	13·8155	16·1181	18·4207	20·7233	23·0259

NAPIERIAN LOGARITHMS (BASE e)

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5·5	1·7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	2	4	5	7	9	11	13	14	16
5·6	1·7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	2	4	5	7	9	11	12	14	16
5·7	1·7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	2	4	5	7	9	10	12	14	16
5·8	1·7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	2	3	5	7	9	10	12	14	15
5·9	1·7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	2	3	5	7	8	10	12	13	15
6·0	1·7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	2	3	5	7	8	10	12	13	15
6·1	1·8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	2	3	5	7	8	10	11	13	15
6·2	1·8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	2	3	5	6	8	10	11	13	14
6·3	1·8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	2	3	5	6	8	10	11	13	14
6·4	1·8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	2	3	5	6	8	9	11	12	14
6·5	1·8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	2	3	5	6	8	9	11	12	14
6·6	1·8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	2	3	5	6	8	9	11	12	14
6·7	1·9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	2	3	4	6	7	9	10	12	13
6·8	1·9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	2	3	4	6	7	9	10	12	13
6·9	1·9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	1	3	4	6	7	9	10	12	13
7·0	1·9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	1	3	4	6	7	9	10	11	13
7·1	1·9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	1	3	4	6	7	8	10	11	13
7·2	1·9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	1	3	4	6	7	8	10	11	12
7·3	1·9879	9892	9906	9920	9933	9947	9961	9974	9988	$\overline{0001}$	1	3	4	5	7	8	10	11	12
7·4	2·0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	1	3	4	5	7	8	9	11	12
7·5	2·0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	1	3	4	5	7	8	9	11	12
7·6	2·0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	1	3	4	5	7	8	9	11	12
7·7	2·0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	1	3	4	5	7	8	9	10	12
7·8	2·0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	1	3	4	5	6	8	9	10	12
7·9	2·0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	1	3	4	5	6	8	9	10	11
8·0	2·0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	1	3	4	5	6	8	9	10	11
8·1	2·0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	1	3	4	5	6	7	9	10	11
8·2	2·1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	1	2	4	5	6	7	9	10	11
8·3	2·1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	1	2	4	5	6	7	8	10	11
8·4	2·1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	1	2	4	5	6	7	8	10	11
8·5	2·1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	1	2	4	5	6	7	8	9	11
8·6	2·1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	1	2	3	5	6	7	8	9	10
8·7	2·1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	1	2	3	5	6	7	8	9	10
8·8	2·1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	1	2	3	5	6	7	8	9	10
8·9	2·1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	1	2	3	5	6	7	8	9	10
9·0	2·1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	1	2	3	4	6	7	8	9	10
9·1	2·2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	1	2	3	4	6	7	8	9	10
9·2	2·2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	1	2	3	4	5	7	8	9	10
9·3	2·2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	1	2	3	4	5	6	8	9	10
9·4	2·2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	1	2	3	4	5	6	7	9	10
9·5	2·2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	1	2	3	4	5	6	7	8	10
9·6	2·2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	1	2	3	4	5	6	7	8	9
9·7	2·2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	1	2	3	4	5	6	7	8	9
9·8	2·2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	1	2	3	4	5	6	7	8	9
9·9	2·2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	1	2	3	4	5	6	7	8	9

NAPIERIAN LOGARITHMS OF 10^{-n}

n	1	2	3	4	5	6	7	8	9	10
$\log_e 10^{-n}$	$\bar{3}$·6974	$\bar{5}$·3948	$\bar{7}$·0922	$\overline{10}$·7897	$\overline{12}$·4871	$\overline{14}$·1845	$\overline{17}$·8819	$\overline{19}$·5793	$\overline{21}$·2767	$\overline{24}$·9741

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
·00	1·0000	1·0000	1·0000	·0000	·40	1·4918	·6703	1·0811	·4108
·01	1·0101	·9901	1·0001	·0100	·41	1·5068	·6636	1·0852	·4216
·02	1·0202	·9802	1·0002	·0200	·42	1·5220	·6570	1·0895	·4325
·03	1·0305	·9704	1·0005	·0300	·43	1·5373	·6505	1·0939	·4434
·04	1·0408	·9608	1·0008	·0400	·44	1·5527	·6440	1·0984	·4543
·05	1·0513	·9512	1·0012	·0500	·45	1·5683	·6376	1·1030	·4653
·06	1·0618	·9418	1·0018	·0600	·46	1·5841	·6313	1·1077	·4764
·07	1·0725	·9324	1·0025	·0701	·47	1·6000	·6250	1·1125	·4875
·08	1·0833	·9231	1·0032	·0801	·48	1·6161	·6188	1·1174	·4986
·09	1·0942	·9139	1·0040	·0901	·49	1·6323	·6126	1·1225	·5098
·10	1·1052	·9048	1·0050	·1002	·50	1·6487	·6065	1·1276	·5211
·11	1·1163	·8958	1·0061	·1102	·51	1·6653	·6005	1·1329	·5324
·12	1·1275	·8869	1·0072	·1203	·52	1·6820	·5945	1·1383	·5438
·13	1·1388	·8781	1·0085	·1304	·53	1·6989	·5886	1·1438	·5552
·14	1·1503	·8694	1·0098	·1405	·54	1·7160	·5827	1·1494	·5666
·15	1·1618	·8607	1·0113	·1506	·55	1·7333	·5769	1·1551	·5782
·16	1·1735	·8521	1·0128	·1607	·56	1·7507	·5712	1·1609	·5897
·17	1·1853	·8437	1·0145	·1708	·57	1·7683	·5655	1·1669	·6014
·18	1·1972	·8353	1·0162	·1810	·58	1·7860	·5599	1·1730	·6131
·19	1·2092	·8270	1·0181	·1911	·59	1·8040	·5543	1·1792	·6248
·20	1·2214	·8187	1·0201	·2013	·60	1·8221	·5488	1·1855	·6367
·21	1·2337	·8106	1·0221	·2115	·61	1·8404	·5434	1·1919	·6485
·22	1·2461	·8025	1·0243	·2218	·62	1·8589	·5379	1·1984	·6605
·23	1·2586	·7945	1·0266	·2320	·63	1·8776	·5326	1·2051	·6725
·24	1·2712	·7866	1·0289	·2423	·64	1·8965	·5273	1·2119	·6846
·25	1·2840	·7788	1·0314	·2526	·65	1·9155	·5220	1·2188	·6967
·26	1·2969	·7711	1·0340	·2629	·66	1·9348	·5169	1·2258	·7090
·27	1·3100	·7634	1·0367	·2733	·67	1·9542	·5117	1·2330	·7213
·28	1·3231	·7558	1·0395	·2837	·68	1·9739	·5066	1·2403	·7336
·29	1·3364	·7483	1·0423	·2941	·69	1·9937	·5016	1·2476	·7461
·30	1·3499	·7408	1·0453	·3045	·70	2·0138	·4966	1·2552	·7586
·31	1·3634	·7334	1·0484	·3150	·71	2·0340	·4916	1·2628	·7712
·32	1·3771	·7261	1·0516	·3255	·72	2·0544	·4868	1·2706	·7838
·33	1·3910	·7189	1·0549	·3360	·73	2·0751	·4819	1·2785	·7966
·34	1·4049	·7118	1·0584	·3466	·74	2·0959	·4771	1·2865	·8094
·35	1·4191	·7047	1·0619	·3572	·75	2·1170	·4724	1·2947	·8223
·36	1·4333	·6977	1·0655	·3678	·76	2·1383	·4677	1·3030	·8353
·37	1·4477	·6907	1·0692	·3785	·77	2·1598	·4630	1·3114	·8484
·38	1·4623	·6839	1·0731	·3892	·78	2·1815	·4584	1·3199	·8615
·39	1·4770	·6771	1·0770	·4000	·79	2·2034	·4538	1·3286	·8748
·40	1·4918	·6703	1·0811	·4108	·80	2·2255	·4493	1·3374	·8881

EXPONENTIAL AND HYPERBOLIC FUNCTIONS (*continued*)

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
·80	2·2255	·4493	1·3374	·8881	1·20	3·3201	·3012	1·8107	1·5095
·81	2·2479	·4449	1·3464	·9015	1·21	3·3535	·2982	1·8258	1·5276
·82	2·2705	·4404	1·3555	·9150	1·22	3·3872	·2952	1·8412	1·5460
·83	2·2933	·4360	1·3647	·9286	1·23	3·4212	·2923	1·8568	1·5645
·84	2·3164	·4317	1·3740	·9423	1·24	3·4556	·2894	1·8725	1·5831
·85	2·3397	·4274	1·3835	·9561	1·25	3·4903	·2865	1·8884	1·6019
·86	2·3632	·4232	1·3932	·9700	1·26	3·5254	·2837	1·9045	1·6209
·87	2·3869	·4190	1·4029	·9840	1·27	3·5609	·2808	1·9208	1·6400
·88	2·4109	·4148	1·4128	·9981	1·28	3·5966	·2780	1·9373	1·6593
·89	2·4351	·4107	1·4229	1·0122	1·29	3·6328	·2753	1·9540	1·6788
·90	2·4596	·4066	1·4331	1·0265	1·30	3·6693	·2725	1·9709	1·6984
·91	2·4843	·4025	1·4434	1·0409	1·31	3·7062	·2698	1·9880	1·7182
·92	2·5093	·3985	1·4539	1·0554	1·32	3·7434	·2671	2·0053	1·7381
·93	2·5345	·3945	1·4645	1·0700	1·33	3·7810	·2645	2·0228	1·7583
·94	2·5600	·3906	1·4753	1·0847	1·34	3·8190	·2618	2·0404	1·7786
·95	2·5857	·3867	1·4862	1·0995	1·35	3·8574	·2592	2·0583	1·7991
·96	2·6117	·3829	1·4973	1·1144	1·36	3·8962	·2567	2·0764	1·8198
·97	2·6379	·3791	1·5085	1·1294	1·37	3·9354	·2541	2·0947	1·8406
·98	2·6645	·3753	1·5199	1·1446	1·38	3·9749	·2516	2·1132	1·8617
·99	2·6912	·3716	1·5314	1·1598	1·39	4·0149	·2491	2·1320	1·8829
1·00	2·7183	·3679	1·5431	1·1752	1·40	4·0552	·2466	2·1509	1·9043
1·01	2·7456	·3642	1·5549	1·1907	1·41	4·0960	·2441	2·1701	1·9259
1·02	2·7732	·3606	1·5669	1·2063	1·42	4·1371	·2417	2·1894	1·9477
1·03	2·8011	·3570	1·5790	1·2220	1·43	4·1787	·2393	2·2090	1·9697
1·04	2·8292	·3535	1·5913	1·2379	1·44	4·2207	·2369	2·2288	1·9919
1·05	2·8577	·3499	1·6038	1·2539	1·45	4·2631	·2346	2·2488	2·0143
1·06	2·8864	·3465	1·6164	1·2700	1·46	4·3060	·2322	2·2691	2·0369
1·07	2·9154	·3430	1·6292	1·2862	1·47	4·3492	·2299	2·2896	2·0597
1·08	2·9447	·3396	1·6421	1·3025	1·48	4·3929	·2276	2·3103	2·0827
1·09	2·9743	·3362	1·6553	1·3190	1·49	4·4371	·2254	2·3312	2·1059
1·10	3·0042	·3329	1·6685	1·3357	1·50	4·4817	·2231	2·3524	2·1293
1·11	3·0344	·3296	1·6820	1·3524	1·51	4·5267	·2209	2·3738	2·1529
1·12	3·0649	·3263	1·6956	1·3693	1·52	4·5722	·2187	2·3955	2·1768
1·13	3·0957	·3230	1·7093	1·3863	1·53	4·6182	·2165	2·4174	2·2008
1·14	3·1268	·3198	1·7233	1·4035	1·54	4·6646	·2144	2·4395	2·2251
1·15	3·1582	·3166	1·7374	1·4208	1·55	4·7115	·2122	2·4619	2·2496
1·16	3·1899	·3135	1·7517	1·4382	1·56	4·7588	·2101	2·4845	2·2743
1·17	3·2220	·3104	1·7662	1·4558	1·57	4·8066	·2080	2·5073	2·2993
1·18	3·2544	·3073	1·7808	1·4735	1·58	4·8550	·2060	2·5305	2·3245
1·19	3·2871	·3042	1·7957	1·4914	1·59	4·9037	·2039	2·5538	2·3499
1·20	3·3201	·3012	1·8107	1·5095	1·60	4·9530	·2019	2·5775	2·3756

[P. T. O.

EXPONENTIAL AND HYPERBOLIC FUNCTIONS *(continued)*

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
1·60	4·9530	·2019	2·5775	2·3756	2·0	7·3891	·1353	3·7622	3·6269
1·61	5·0028	·1999	2·6013	2·4015	2·1	8·1662	·1225	4·1443	4·0219
1·62	5·0531	·1979	2·6255	2·4276	2·2	9·0250	·1108	4·5679	4·4571
1·63	5·1039	·1959	2·6499	2·4540	2·3	9·9742	·1003	5·0372	4·9370
1·64	5·1552	·1940	2·6746	2·4806	2·4	11·023	·0907	5·5570	5·4662
1·65	5·2070	·1921	2·6995	2·5075	2·5	12·183	·0821	6·1323	6·0502
1·66	5·2593	·1901	2·7247	2·5346	2·6	13·464	·0743	6·7690	6·6947
1·67	5·3122	·1882	2·7502	2·5620	2·7	14·880	·0672	7·4735	7·4063
1·68	5·3656	·1864	2·7760	2·5896	2·8	16·445	·0608	8·2527	8·1919
1·69	5·4195	·1845	2·8020	2·6175	2·9	18·174	·0550	9·1146	9·0596
1·70	5·4739	·1827	2·8283	2·6456	3·0	20·086	·0498	10·068	10·018
1·71	5·5290	·1809	2·8549	2·6740	3·1	22·198	·0450	11·122	11·076
1·72	5·5845	·1791	2·8818	2·7027	3·2	24·533	·0408	12·287	12·246
1·73	5·6407	·1773	2·9090	2·7317	3·3	27·113	·0369	13·575	13·538
1·74	5·6973	·1755	2·9364	2·7609	3·4	29·964	·0334	14·999	14·965
1·75	5·7546	·1738	2·9642	2·7904	3·5	33·116	·0302	16·573	16·543
1·76	5·8124	·1720	2·9922	2·8202	3·6	36·598	·0273	18·313	18·285
1·77	5·8709	·1703	3·0206	2·8503	3·7	40·447	·0247	20·236	20·211
1·78	5·9299	·1686	3·0493	2·8806	3·8	44·701	·0224	22·362	22·339
1·79	5·9895	·1670	3·0782	2·9112	3·9	49·402	·0202	24·711	24·691
1·80	6·0496	·1653	3·1075	2·9422	4·0	54·598	·0183	27·308	27·290
1·81	6·1104	·1637	3·1370	2·9734	4·1	60·340	·0166	30·178	30·162
1·82	6·1719	·1620	3·1669	3·0049	4·2	66·686	·0150	33·351	33·336
1·83	6·2339	·1604	3·1971	3·0367	4·3	73·700	·0136	36·857	36·843
1·84	6·2965	·1588	3·2277	3·0689	4·4	81·451	·0123	40·732	40·719
1·85	6·3598	·1572	3·2585	3·1013	4·5	90·017	·0111	45·014	45·003
1·86	6·4237	·1557	3·2897	3·1340	4·6	99·484	·0101	49·747	49·737
1·87	6·4883	·1541	3·3212	3·1671	4·7	109·95	·0091	54·978	54·969
1·88	6·5535	·1526	3·3530	3·2005	4·8	121·51	·0082	60·759	60·751
1·89	6·6194	·1511	3·3852	3·2342	4·9	134·29	·0074	67·149	67·141
1·90	6·6859	·1496	3·4177	3·2682	5·0	148·41	·0067	74·210	74·203
1·91	6·7531	·1481	3·4506	3·3025	5·1	164·02	·0061	82·014	82·008
1·92	6·8210	·1466	3·4838	3·3372	5·2	181·27	·0055	90·639	90·633
1·93	6·8895	·1451	3·5173	3·3722	5·3	200·34	·0050	100·17	100·17
1·94	6·9588	·1437	3·5512	3·4075	5·4	221·41	·0045	110·71	110·70
1·95	7·0287	·1423	3·5855	3·4432	5·5	244·69	·0041	122·35	122·34
1·96	7·0993	·1409	3·6201	3·4792	5·6	270·43	·0037	135·21	135·21
1·97	7·1707	·1395	3·6551	3·5156	5·7	298·87	·0033	149·44	149·43
1·98	7·2427	·1381	3·6904	3·5523	5·8	330·30	·0030	165·15	165·15
1·99	7·3155	·1367	3·7261	3·5894	5·9	365·04	·0027	182·52	182·52
2·00	7·3891	·1353	3·7622	3·6269	6·0	403·43	·0025	201·72	201·71

ANSWERS

PART II

EXAMPLES XI a (p. 150)

1. $15x^2+2$. 2. $20x^4-20x^3$. 3. $15x^5-5x^3$. 4. $2x+5$. 5. $6x^2-2x-1$.
6. $10x^4+20x^3+9x^2+22x+16$. 7. $3x^2+12x+11$. 8. $8x^3-3x^2+2$.
9. $\dfrac{1}{(x+2)^2}$. 10. $\dfrac{2-2x}{x^3}$. 11. $-\dfrac{29}{(5x-3)^2}$. 12. $\dfrac{2x}{(x^2+1)^2}$.
13. $\dfrac{(x+1)(x-3)}{(x-1)^2}$. 14. $\dfrac{2-2x}{(x+1)^3}$. 15. $\dfrac{3x^2-4x+19}{(3x-2)^2}$. 16. $10(2x+3)^4$.
17. $-24(4-3x)^7$. 18. $-12x(2-x^2)^5$. 19. $(x+2)(3x-4)$.
20. $\dfrac{1+x}{(1-x)^3}$. 21. $2(ax+b)(cx+d)(2acx+bc+ad)$. 22. $\dfrac{(5x-3)^2(42-5x)}{(x+2)^5}$.
23. $\dfrac{2(2x-3)(3-7x)}{(2x-3x^2)^2}$. 24. $2(3x^2+1)(x^3+x-5)$. 25. $\dfrac{1}{(x+2)^2}-\dfrac{1}{(x+1)^2}$.
26. $-\dfrac{4x}{(x^2+4)^3}$. 27. $\dfrac{4(x-1)}{(x+1)^3}$.

EXAMPLES XI b (p. 152)

1. $-\dfrac{3}{x^4},\ 2x+\dfrac{4}{x^3},\ -\dfrac{4}{x^2}-\dfrac{15}{x^4},\ -\dfrac{100}{x^{11}},\ -\dfrac{2n}{x^{2n+1}}$.

2. $1{\cdot}5x^{\frac{1}{2}},\ \frac{2}{3}x^{-\frac{1}{3}},\ \frac{1}{3}x^{-\frac{2}{3}},\ \frac{9}{2}x^{\frac{1}{2}},\ \frac{5}{4}x^{\frac{3}{2}},\ \dfrac{1}{q}x^{\frac{1}{q}-1},\ \dfrac{2}{n}x^{\frac{2}{n}-1}$.

3. $-\frac{1}{2}x^{-\frac{3}{2}},\ -3x^{-4},\ -\frac{1}{3}x^{-\frac{4}{3}},\ 0,\ -\frac{4}{3}x^{-\frac{4}{3}},\ -\dfrac{1}{q}x^{-\frac{1}{q}-1}$.

4. $-2{\cdot}3x^{-3{\cdot}3},\ -6x^{-3},\ -\frac{3}{2}\sqrt{5}\,x^{-2{\cdot}5},\ x^{-\frac{2}{3}},\ -2x^{-\frac{3}{2}},\ -3x^{-\frac{5}{2}},\ -\dfrac{1}{\sqrt{(5-2x)}}$.

5. $3x^2+2+\dfrac{3}{x^2}+\dfrac{18}{x^4}$. 6. $3nx^{3n-1}+2nx^{2n-1}-3nx^{n-1}$. 7. $\dfrac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}$.

8. $\frac{1}{2}(3-x)^{-1{\cdot}5}$. 9. $(x-1)(x^2-2x)^{-\frac{1}{2}}$. 10. $\dfrac{x}{(9-x^2)^{\frac{3}{2}}}$. 11. $3x^2(3x^3+8)^{-\frac{2}{3}}$.

12. $-(x^2-1)^{-1{\cdot}5}$. 13. $\dfrac{2x-3x^3}{\sqrt{(1-x^2)}}$. 14. $\frac{3}{2}(1-x)^{-\frac{3}{2}}(1+2x)^{-\frac{1}{2}}$.

15. $1+\dfrac{x}{\sqrt{(1+x^2)}}$. 16. $\dfrac{3[x+\sqrt{(1+x^2)}]^3}{\sqrt{(1+x^2)}}$. 17. $\dfrac{5x+9}{2(1+x)^2\sqrt{(2x^2-x-5)}}$.

18. $\dfrac{1}{2}\left(\dfrac{1}{\sqrt{(x+1)}}+\dfrac{1}{\sqrt{x}}\right)$. 19. $\mp x^{-\frac{3}{2}},\ \mp x^{\frac{3}{2}},\ 1$. 26. $0{\cdot}013$ sec.

27. Min., $\frac{27}{32}$; inflexion, 0. 28. $\frac{500}{121}$, $\frac{5}{4}$ ft. sec. 29. Max., $x=2$; min., $x=\frac{48}{19}$.
30. a^2. 31. 20 ft. from A. 32. $\frac{3}{2}\sqrt{2}=2{\cdot}12$ ft. 33. $\frac{6}{7}\sqrt{7}=2{\cdot}27$ m.

EXAMPLES XI c (p. 156)

1. $-\dfrac{x}{y}$. 2. $\dfrac{2x}{3y}$. 3. $-\dfrac{y^2}{x^2}$. 4. $\dfrac{y-x^2}{y^2-x}$. 5. $-\dfrac{2xy+y^2}{x^2+2xy}$. 6. $-\dfrac{3y}{2x}$.
7. $\dfrac{y+2}{3-x}$. 8. $-\sqrt{\left(\dfrac{y}{x}\right)}$. 9. $\dfrac{1}{t}$. 10. $\dfrac{t-1}{t}$. 11. $\dfrac{m(2-m^3)}{1-2m^3}$. 12. $\dfrac{2t}{t^2-1}$.

13. 4 in. sec. **14.** 41° 59′ with horiz. **15.** 2·5 ft. sec. **17.** $x+y$.

18. $-\sqrt[3]{\left(\dfrac{y}{x}\right)}$. **19.** 4. **20.** 3 ft. sec. **21.** 4 in. min. **23.** 1·09 ft. sec.

25. $0{\cdot}166l$ ft. sec. **26.** 1; $-\dfrac{4}{3at^4}$. **27.** 3; 4·8 in. sec.

MISCELLANEOUS EXAMPLES M. 13—17 (p. 158)

M. 13. 2. $\sqrt{(625t^2-1200t+1600)}$; 0·96. **4.** 5·985 sq. ft. sec.

5. Least and min., $x=\dfrac{50}{8+\pi}$; greatest not max., $x=\dfrac{50}{\pi}$.

M. 14. 2. $\sqrt{(bc)}$. **3.** 1. **4.** 150π sq. ft. sec. **5.** $3\sqrt{2}$.

M. 15. 1. $12y=x$, $x+6y=1$, $x+5y+1=0$. **4.** $h=\dfrac{\lambda-\gamma}{\lambda-1}\cdot p$.

M. 16. 1. $p_2=0{\cdot}53p_1$. **2.** $\theta^2=\dfrac{8c}{27(a+b)}$. **5.** 3·14 atmos.

M. 17. 1. length of chord. **2.** $\dfrac{8x+5x^2}{2\sqrt{(2+x)}}$; 0 at $x=0$, ∞ at $x=-2$; $x=-1{\cdot}6$; 123° 41′. **3.** $\dfrac{2A}{k}\sqrt{c}$. **4.** $-0{\cdot}0024$; 0·7676. **5.** $-s$ ft. sec.^{2}.

EXAMPLES XII a (p. 169)

1. $3\cos(3x+4)$, $-3\sin 3x$, $-\cos(2-x)$. **2.** $4\sec^2 4(x-2)$, $\sin(a-x)$, $\dfrac{2\pi}{3}\cos\dfrac{2\pi}{3}(x+4)$. **3.** $2\sec 2x\tan 2x$, $-\sec(3-x)\tan(3-x)$, $4\sec^2(4x-3)$.

4. $-4\operatorname{cosec} 4x\cot 4x$, $\dfrac{2\pi}{a}\cos\dfrac{2\pi}{a}(x-b)$, $-3\operatorname{cosec}(3x-1)\cot(3x-1)$.

5. $-a\operatorname{cosec}^2 ax$, $\operatorname{cosec}^2(2-x)$, $-3\sec^2 3(2-x)$.

6. $\sin 2x$, $-2\sin 4x$, $2\tan(x-2)\sec^2(x-2)$.

7. $3\sin(6x-8)$, $-9\cos^2 3x\sin 3x$, $-2\operatorname{cosec}^2(x-2)\cot(x-2)$.

8. $2a\sec^2(ax-b)\tan(ax-b)$, $-6\operatorname{cosec}^3 2x\cot 2x$, $-\frac{1}{2}x^{-\frac{1}{2}}\operatorname{cosec}^2(x^{\frac{1}{2}})$.

9. $\sin x+x\cos x$, $2x\cos x-x^2\sin x$, $x\cos x$.

10. $\tan^2 x$, $\tan x+x\sec^2 x$, $\cos x\cos 2x-2\sin x\sin 2x$.

11. $\cos x\sin 2x+2\sin x\cos 2x$, $a\cos ax\cos bx-b\sin ax\sin bx$, $-p\sin px\cos qx-q\cos px\sin qx$.

12. $\cos^2 x$, $2\sec^2 2x\cos x-\tan 2x\sin x$, $2x\operatorname{cosec} x-x^2\operatorname{cosec} x\cot x$.

13. $\dfrac{x\cos x-\sin x}{x^2}$, $\dfrac{-2\cos x}{(1+\sin x)^2}$, $-\dfrac{\cos x}{(1+\sin x)^2}$.

14. $\cos x+\sin x$, $\dfrac{a}{x^2}\sin\dfrac{a}{x}$, $\dfrac{\cos 2x}{\sqrt{(\sin 2x)}}$.

15. $\sec^2\dfrac{x}{2}\tan\dfrac{x}{2}$, $\sec^2\dfrac{x}{2}\tan\dfrac{x}{2}$, $-\dfrac{1}{1+\sin 2x}$.

16. $\dfrac{\pi}{90}\cos(2x^\circ)$, $-\dfrac{\pi}{180}\sin(2x^\circ)$, $\dfrac{\pi}{360}\sec^2(\frac{1}{2}x^\circ)$.

EXAMPLES XII b (p. 171)

1. $\frac{1}{2}\sin 2x+c$; $-\frac{1}{3}\cos 3x+c$. 2. $\frac{1}{2}\tan 2x+c$; $\frac{1}{2}\tan 2x-x+c$.

3. $-\dfrac{\cos(ax+bx)}{2(a+b)}-\dfrac{\cos(ax-bx)}{2(a-b)}+c$; $\dfrac{x}{2}+\frac{1}{4}\sin 2x+c$. 7. 5, −5.

8. $\dfrac{2\sqrt{3}}{3}$ ft. sec. 9. 0·0174, 0·0164, 0·0133, 0·0088.

10. $-a\omega\sin(\omega t)$; $-a\omega^2\cos(\omega t)$.

15. $\dfrac{d\theta}{dt}=-\dfrac{v}{2a}\cos\theta\cot\theta$; $\dfrac{v}{2}\operatorname{cosec}\theta$; $-\dfrac{v}{2a\sqrt{2}}$; $\dfrac{v}{2}\sqrt{2}$.

16. $\dfrac{\pi\sqrt{2}}{180}=0{\cdot}0247$ ft. sec.; 0·0247 ft. sec.

EXAMPLES XII c (p. 175)

1. $\dfrac{2}{\sqrt{(1-4x^2)}}$. 2. $-\dfrac{3}{\sqrt{(1-9x^2)}}$. 3. $\dfrac{4}{1+16x^2}$. 4. $\dfrac{1}{\sqrt{(9-x^2)}}$.

5. $\dfrac{1}{x\sqrt{(1-x^2)}}-\dfrac{1}{x^2}\sin^{-1}x$. 6. $2x\cos^{-1}x-\dfrac{x^2}{\sqrt{(1-x^2)}}$. 7. $-\dfrac{a}{x\sqrt{(x^2-a^2)}}$.

8. $\dfrac{a}{x\sqrt{(x^2-a^2)}}$. 9. $-\dfrac{a}{x^2+a^2}$. 10. 0. 11. $2x\tan^{-1}x$.

12. $\dfrac{1}{\sqrt{(1-x^2)}}$. 13. $\dfrac{1}{a}$. 14. 1. 15. $\frac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+c$.

16. $\sin^{-1}\left(\dfrac{x}{3}\right)+c$. 17. $\frac{1}{3}\tan^{-1}\left(\dfrac{x+1}{3}\right)+c$. 18. $\cos^{-1}x$.

REVISION PAPERS R. 12—17 (p. 176)

R. 12. 2. $\frac{8}{15}$. 3. $\dfrac{x-1}{2x\sqrt{x}}$; $\dfrac{n}{2}x^{n-1}(x^n+2)^{-\frac{1}{2}}$; $\dfrac{1}{\cos x-1}$.

4. 3. 5. $\frac{1}{2}\sin x-\frac{1}{10}\sin 5x+c$; $\frac{1}{2}$.

R. 13. 2. 500 cu. ft. 3. $(\frac{3}{7}, 0)$. 4. $\dfrac{3x^2}{(4-x^3)^2}$; $1-\dfrac{x}{\sqrt{(1+x^2)}}$;

$3\sec^3 x\tan x$. 5. $-\frac{2}{3}x^{-\frac{4}{3}}$; $-\frac{3}{2}x^{\frac{4}{3}}$; 1.

R. 14. 1. $\dfrac{10}{3}\sqrt{\left(\dfrac{10}{3\pi}\right)}=3{\cdot}43$ cu. in. 2. $\dfrac{4096\pi}{15}=858$.

3. $\dfrac{x^2-4x-1}{(x^2+1)^2}$; $\dfrac{1}{x\sqrt{(4x^2-1)}}$; $-\dfrac{\pi}{60}\sin(3x^\circ)$.

4. $\pi\sqrt{3}=5{\cdot}44$ in. sec. 5. 2220.

R. 15. 1. $\dfrac{2t}{t^2-1}$. 2. −0·05. 3. $\dfrac{1}{2\sqrt{x}}\cos(\sqrt{x})$; $\dfrac{\cos x}{2\sqrt{(\sin x)}}$.

4. $\frac{9}{128}$. 5. $10(\sqrt{2}-1)=4{\cdot}14$ in.

R. 16. 1. $x=\dfrac{3t}{1+t^3}$, $y=\dfrac{3t^2}{1+t^3}$; (i) $t=0$ or $\sqrt[3]{2}$; (ii) $t=\infty$ or $\dfrac{1}{\sqrt[3]{2}}$.

2. $\dfrac{x^{n-1}}{\lfloor n-1}$; 0; $a^4\sin ax$. 3. $2a\omega\sin(\frac{1}{2}\omega t)$.

4. $\sqrt{(\frac{541}{48})}=3{\cdot}36$ in.; $\frac{4}{3}\sqrt{3}=2{\cdot}31$ in. 5. $\sqrt{\left(\dfrac{3b}{a}\right)}$.

R. 17. **1.** $\sec x\sqrt{(\sec 2x)}$. **2.** 0·084 per cent.
3. $\frac{1}{20}$; 0·89; $-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$. **4.** $29\frac{1}{6}$ in. lb. **5.** 4·95 sq. ft.

MISCELLANEOUS EXAMPLES M. 18 23 (p. 179)

M. 18. **1.** $\frac{52\pi}{9}=18·1$ cm. sec. **2.** $2a\sin^2\theta\sin 2\theta$; 60°.
3. 34 7. **4.** $-\omega^2V$; $-n^2V$; $\frac{\omega^2}{n^2}$.

M. 19. **1.** 0·946; $-12·65$ ft. sec.^{2}. **3.** 2·00101.
5. $a^2\omega\sec^2\theta$; $ah\omega\,\mathrm{cosec}^2\,\theta$ sq. ft. sec.

M. 20. **1.** $-\frac{b}{a}\cot\theta$. **2.** 60°. **3.** $-8·41$. **5.** $\theta=\lambda$.

M. 21. **1.** $(-2-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$; $\frac{\pi}{4}\sqrt{(5-2\sqrt{2})}=1·16$ ft. sec.; $\frac{\pi^2}{16}=0·617$ ft. sec.^{2}.
2. 0·1 rad. sec. **3.** $\frac{1}{\pi}(\pi-2\theta+\sin 2\theta)$; $-2a^2(1-\cos 2\theta)$.
4. $ae>bc$. **5.** $52·8\tan^2\theta$ ft. sec.

M. 22. **2.** 9·27 ft. **4.** $-\omega x\tan\theta$.
5. $2l\sin\alpha\sin(\beta-\theta)\,\mathrm{cosec}(\alpha+\beta)+c\sin\theta$; $\tan\theta=\frac{c\cot\alpha-(2l-c)\cot\beta}{2l}$.

M. 23. **1.** Lose 2·2 sec. per day. **2.** $1:\sqrt{3}$.

EXAMPLES XIII a (p. 188)

1. $2e^{2x}$. **2.** e^x-e^{-x}. **3.** e^x+e^{-x}. **4.** $2x\,.\,e^{x^2}$. **5.** $a\,.\,e^{ax+b}$.
6. $e^x(x+1)$. **7.** $e^{3x}(3\sin 2x+2\cos 2x)$. **8.** $-e^{-2x}(2\cos 3x+3\sin 3x)$.
9. $e^{ax}\,\mathrm{cosec}^2\,bx\,(a\sin bx-b\cos bx)$. **10.** $\frac{1}{x}$. **11.** $\frac{1}{x-1}$. **12.** $\frac{3}{x}$.
13. $-\tan x$. **14.** $\sec x\,\mathrm{cosec}\,x$. **15.** $\frac{1+\cos x}{x+\sin x}$. **16.** $2x\log x+x$.
17. $\frac{2a}{x^2-a^2}$. **18.** $\mathrm{cosec}\,x$. **19.** $\sec x$. **20.** $2\sec x$. **21.** $\frac{1}{2}e^{2x}+c$.
22. $-\frac{1}{3}e^{-3x}+c$. **23.** $\frac{1}{a}e^{ax+b}+c$. **24.** $5\log x+c$. **25.** $-\frac{3}{5}\log(4-5x)+c$.
26. $-\log(\cos x)+c$. **27.** $\frac{1}{3}\log(\sin 3x)+c$. **28.** $\log(1+\sin x)+c$.
29. $\log\left(\frac{x+2}{x+3}\right)+c$. **30.** $1+\log x$; $x\log x-x+c$.
31. 2·92, 2·75, 2·59, 2·44, 2·3, 2·3. **32.** 0·7077. **36.** 3 or -5.

EXAMPLES XIII b (p. 193)

1. $\log x$. **2.** $3e^{2x}$. **3.** $2e^{-3x}$. **4.** Gradient at P equals y. **5.** $e-1$.
6. 1. **7.** Max. at (0, 1). **8.** $1-e^{-\frac{T}{\lambda}}$. **11.** $1, -e^{-\frac{\pi}{2}}=-0·208$.
12. 1; b. **13.** 78·4 per cent.
14. $0·1<x<1$ more rapidly, $1<x<10$ less rapidly. **15.** $5e^2=36·9$ ft. sec.^{2}.

EXAMPLES XIII c (p. 196)

1. $10^x \log 10$; $\dfrac{1}{x \log 10}$; $-3^{-x} \log 3$; $\dfrac{2x}{(1+x^2)\log 10}$; $-\dfrac{\tan x}{\log 10}$; $\dfrac{2^x}{\log 10}\left(\log x \log 2 + \dfrac{1}{x}\right)$.
2. $\dfrac{10^x}{\log 10}+c$; $\dfrac{3^{2x}}{\log 9}+c$; $\dfrac{a^{cx}}{c \log a}+b$.
3. 0·41; 39; 3·2.
6. ac^3e^{cx}; $\dfrac{2}{x^3}$.
11. $p=\dfrac{a}{a^2+b^2}$; $q=-\dfrac{b}{a^2+b^2}$.
12. $s=\dfrac{c}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right)$.
13. 21·1 tons sq. in.
14. $e^{0\cdot182x}$; 0·378; 5·89.

EXAMPLES XIII d (p. 202)

1. $\dfrac{u}{k}(e^{kt}-1)$ ft.
2. 30·5°.
3. 6·93; 0·76 sec.
4. 1710 ft. sec.
5. 0·354.
6. 1007 cu. cm.
9. 55·7°.
10. 2·75.
11. $\omega_1\left(\dfrac{\omega_2}{\omega_1}\right)^t$.
12. 31·6 min.
13. $2g$; $4g\left(4+\dfrac{1}{e^5}\right)=16{\cdot}03g$.
14. $\dfrac{b}{n}$; $n<0$.
15. $A=0$, $B=T$.
16. $C=C_0e^{-kt}$; $p=p_0e^{klt}$; $p\,.\,C^l=p_0\,.\,C_0{}^l$.
17. $2\frac{1}{2}$; 2·69; 3·37 sq. in.
18. 333 ft. sec.
19. $-\dfrac{V}{R}$.
20. 135 ft. lb.

EXAMPLES XIV a (p. 208)

1. (i) $2ax\,dx$; (ii) $-4x\,dx$; (iii) $(6x-5)\,dx$; (iv) dx; (v) $\frac{1}{2}x^{-\frac{1}{2}}dx$; (vi) $-\frac{1}{2}x^{-\frac{3}{2}}dx$; (vii) $\left(1-\dfrac{1}{x^2}\right)dx$; (viii) $-\sin\theta\,d\theta$; (ix) $\sec\theta\tan\theta\,d\theta$; (x) $2\sec^2(2x+3)\,dx$; (xi) $\dfrac{dx}{1+x^2}$; (xii) $2\sec^2\theta\tan\theta\,d\theta$; (xiii) $3\cos(3x+5)\,dx$; (xiv) $-\operatorname{cosec}(2a+x)\cot(2a+x)\,dx$; (xv) $-\sec(a-x)\tan(a-x)\,dx$; (xvi) $\dfrac{dx}{x}$; (xvii) $\dfrac{a\,dx}{ax+b}$; (xviii) $\frac{1}{2}(x^{-\frac{1}{2}}-3x^{-\frac{5}{2}})\,dx$; (xix) $a\,.\,e^{ax}dx$; (xx) $-3e^{-3x}dx$.
2. $x+c$; x^2+3x+c; $2\sqrt{x}+c$; $-\dfrac{5}{2v^{0\cdot4}}+c$; $-2u^{-\frac{1}{2}}+c$; $c+\log(u+1)$; $x+\dfrac{3}{x}+c$; $c+\sec\theta$; $c-\cot\theta$.
3. 0·0087.
4. 0·10 per cent.
5. 0·7 per cent.
6. $(a+b+c)$ per cent.
7. $\dfrac{32\sqrt{3}}{9}=6{\cdot}16$ sq. in.
8. Height = diameter.
9. A and B acute, 2 per cent.; A or B obtuse, $\dfrac{2(a^2\sim b^2)}{c^2}$ per cent.
10. 3·4°.
12. 4·14 in. sec.
13. 0·5 per cent.

EXAMPLES XIV b (p. 210)

1. $\frac{1}{2}(4W+w)\tan\alpha$.
2. $W\tan\theta$.
3. $\dfrac{\sqrt{6}}{6}=0{\cdot}41$ lb.; $\dfrac{7\sqrt{6}}{18}=0{\cdot}95$ lb.
4. $W\sqrt{3}$.

EXAMPLES XIV c (p. 212)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{3}(x+4)^3$. 2. $-\frac{1}{4}(2-x)^4$. 3. $\frac{1}{3}(2x+3)^{\frac{3}{2}}$. 4. $-\dfrac{1}{4(2x+1)^2}$.

5. $\frac{1}{3}(2x+5)^{\frac{3}{2}}$. 6. $\dfrac{2}{3a}(ax+b)^{\frac{3}{2}}$. 7. $\frac{1}{2}\sin(2x-5)$. 8. $\cos(4-x)$.

9. $\dfrac{1}{n}\tan(nx+a)$. 10. $\frac{1}{3}\sin^3 x$. 11. $\frac{1}{3}\sin^3 x$. 12. $\frac{1}{2}\tan^2 x$.

13. $\frac{1}{2}\tan^2 x$. 14. $-\frac{1}{2}\cot^2 x$. 15. $-\dfrac{1}{a}e^{-ax}$. 16. $\dfrac{1}{a}e^{ax+b}$.

17. $\frac{1}{2}\log(2x-3)$. 18. $\dfrac{1}{a}\log(ax+b)$. 19. $-\log(2-x)$. 20. $\frac{5}{7}x^{1\cdot4}$.

21. $-0\cdot6055$. 22. $\frac{5}{3}x^{0\cdot6}$. 23. $\frac{1}{3}(x^2+2x+3)^{\frac{3}{2}}$.

24. $\dfrac{1}{2(1+2\cos x)}$. 25. $-\frac{1}{2}e^{-x^2}$.

EXAMPLES XIV d (p. 215)

[*Note:* the arbitrary constant is omitted.]

1. $-\frac{1}{3}(3-x^2)^{\frac{3}{2}}$. 2. $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$. 3. $-2\sqrt{(1-x)}$. 4. $-\sqrt{(1-x^2)}$.

5. $\frac{2}{3}\sqrt{(a^2+x^3)}$. 6. $\frac{2}{3}\pi\sqrt{(y^2+a^2)^3}$. 7. $0\cdot828$. 8. $0\cdot490$.

9. $\frac{1}{3}\log(3+t^3)$. 10. $\dfrac{3}{2a}\log(b^2+a^2x^2)$. 11. $-\log(3-x)$.

12. $-\dfrac{1}{b}\log(a-bx)$. 13. $\log(\sin x)$. 14. $\log(x^3-x^2+1)$.

15. $\frac{1}{2}(\log x)^2$. 16. $\frac{1}{6}(1+x^4)^{\frac{3}{2}}$. 17. $\frac{1}{4}\sin^4 x$. 18. $0\cdot841$. 19. $\frac{1}{6}\tan^6\theta$.

EXAMPLES XIV e (p. 219)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{2}x+\frac{1}{4}\sin 2x$. 2. $\frac{1}{2}x+\frac{1}{12}\sin 6x$. 3. $-\dfrac{\cos(ax+bx)}{2(a+b)}-\dfrac{\cos(ax-bx)}{2(a-b)}$.

4. $\tan x - x$. 5. $\cos(1-x)$. 6. $0\cdot123$. 7. $\frac{1}{2}T$.

8. $\frac{1}{12}(\cos 3x - 9\cos x)$. 9. $\dfrac{\pi}{32}$. 10. $\sin^{-1}\left(\dfrac{x}{4}\right)$. 11. π.

12. $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)$. 13. $\dfrac{\pi}{4a}$. 14. $\tan^{-1}(x+3)$.

15. $-\sin^{-1}\left(\dfrac{1-x}{2}\right)$. 16. $\dfrac{7\sqrt{3}}{6}\sin^{-1}(x\sqrt{\frac{3}{7}})+\dfrac{x}{2}\sqrt{(7-3x^2)}$. 17. $3\cdot55$.

18. $-\sin^{-1}\left(\dfrac{a-2x}{a}\right)$. 19. $\sin^{-1}(x-3)$. 20. $-x-\dfrac{1}{a}\cot(ax+b)$.

21. $\dfrac{3\pi}{16}$. 24. $\dfrac{\pi}{16}$. 25. $\dfrac{\pi}{4}$. 26. k; $\dfrac{2k}{\pi}$.

EXAMPLES XIV f (p. 223)

[*Note:* the arbitrary constant is omitted.]

1. $x - \log(1+x)$.
2. $x - \log(x+3)$.
3. $\frac{1}{2}x^2 - 2x + 7\log(x+2)$.
4. $\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$.
5. $\frac{1}{12}\log\left(\frac{2x-3}{2x+3}\right)$.
6. $\frac{1}{2}\log(x^2-4)$.
7. $\frac{1}{2}\tan^{-1}(2x)$.
8. $\frac{3}{8}\log(4x^2+1) + \tan^{-1}(2x)$.
9. $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}\log(x^2+16)$.
10. $\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)$.
11. $\frac{5}{2}\log(x^2-2x+10) + \frac{5}{3}\tan^{-1}\left(\frac{x-1}{3}\right)$.
12. $\frac{1}{2}\log[(x+a)^2+b^2] + \frac{1-a}{b}\tan^{-1}\left(\frac{x+a}{b}\right)$.
13. $\log\left(\frac{x+1}{x+2}\right)$.
14. $4\log(x-3) - 3\log(x-2)$.
15. $-\log[(1-x)^2(2+x)^3]$.
16. $-\log[(5+x)^2(2-x)]$.
17. $\frac{1}{x} + \log\left(1-\frac{1}{x}\right)$.
18. $\log\left(\frac{x+2}{x-1}\right) - \frac{1}{x-1}$.
19. $\frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x - 3\log(x+2)$.
20. $3\log(x-2) - \frac{3}{2}\log(x^2+8x+17) + 12\tan^{-1}(x+4)$.
21. $\frac{1}{11}\log(x-2) - \frac{1}{22}\log(x^2+3x+1) - \frac{7}{22\sqrt{5}}\log\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}$.
22. $2\log(x-2) - \frac{3}{2}\log(x-1) - \frac{1}{2}\log(x-3)$.
23. $\log(x+1) + \frac{2}{x+1} - \frac{1}{2(x+1)^2}$.
24. $\log x - \frac{1}{2}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$.
25. 0·307.
26. $\frac{\pi\sqrt{3}}{9}$.
27. 8·27.
28. $\log\left(\frac{2a}{a+1}\right) + \frac{1-a}{2(1+a)}$; 0·193.
29. $y = x + 1 - 2\log(x+2)$.
30. $\frac{1}{2kV}\log 3$.

EXAMPLES XIV g (p. 226)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{4}(2x^2\log x - x^2)$.
2. $e^x(x-1)$.
3. $\frac{1}{9}(3x^3\log x - x^3)$.
4. $\sin x - x\cos x$.
5. $\frac{x(1+x)^{11}}{11} - \frac{(1+x)^{12}}{132}$.
6. $\frac{1}{9}(3x\sin 3x + \cos x)$.
7. $x\tan x + \log\cos x$.
8. $\frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x$.
9. $\frac{1}{4}(\sin 2x - 2x\cos 2x)$.
10. $x\sin^{-1}x + \sqrt{(1-x^2)}$.
11. $(x^2-2)\sin x + 2x\cos x$.
12. $-\frac{1+2\log x}{4x^2}$.
13. $\frac{1}{2}e^x(\sin x - \cos x)$.
14. $\frac{1}{2}e^x(\sin x + \cos x)$.
15. $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$.
16. $\frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1]$.
17. $-\frac{1}{13}e^{-2x}(2\sin 3x + 3\cos 3x)$.
18. $-\frac{(1-x)^{21}(231x^2+21x+1)}{5313}$.
20. $-x^5\cos x + 5x^4\sin x + 20x^3\cos x - 60x^2\sin x - 120x\cos x + 120\sin x$.
21. $\frac{35\pi}{256}$.
23. Integral $= -x^n e^{-x} + n\int x^{n-1}e^{-x}dx$; $\lfloor 10$.
24. $\frac{3\pi}{512}$; $\frac{16}{1155}$.

EXAMPLES XIV h (p. 231)

[*Note:* the arbitrary constant is omitted.]

1. $2\log\left(\tan\frac{\theta}{4}\right)$. 2. $2\log\left[\tan\frac{\pi+\theta}{4}\right]$. 3. $\log\tan\theta$.

4. $\frac{1}{\sqrt{2}}\log\left[\tan\frac{4x+\pi}{8}\right]$. 5. $\frac{1}{2}\tan^{-1}\left[\frac{1}{2}\tan\frac{\theta}{2}\right]$. 6. $\frac{1}{4}\log\left(\dfrac{2+\tan\frac{\theta}{2}}{2-\tan\frac{\theta}{2}}\right)$.

7. $\log\left(\frac{\sqrt{5}+2}{\sqrt{2}+1}\right)$. 8. $\frac{1}{2}[\sqrt{2}+\log(1+\sqrt{2})]$. 9. $\frac{\pi}{\sqrt{5}}$.

10. $\frac{x}{4}(1+x^2)^{\frac{3}{2}}+\frac{3x}{8}\sqrt{(1+x^2)}+\frac{3}{8}\log[x+\sqrt{(1+x^2)}]$.

11. $\frac{x}{2}\sqrt{(x^2-16)}-8\log[x+\sqrt{(x^2-16)}]$.

12. $\frac{1}{4}(2x-3)\sqrt{(x^2-3x-10)}-\frac{49}{8}\log[x-\frac{3}{2}+\sqrt{(x^2-3x-10)}]$.

13. $\frac{8}{15}$. 14. $\frac{5\pi}{32}$. 15. $\frac{2}{143}$. 16. $\frac{5\pi}{256}$. 17. $\frac{5\pi}{32}$. 18. $\frac{16}{1155}$. 19. $\frac{1}{280}$.

20. $\frac{\pi a^4}{16}$. 21. $\frac{\pi}{2}$. 22. $\frac{\pi}{8}$. 23. $\frac{3\pi a^2}{16}$. 24. $\frac{3\pi}{16}$. 25. $\frac{3\pi}{8}$.

EXAMPLES XIV i (p. 232)

[*Note:* the arbitrary constant is omitted.]

1. $-\frac{x^{-0\cdot37}}{0\cdot37}$. 2. $\frac{3}{4}(a+x)^{\frac{4}{3}}$. 3. $\tan^{-1}x$. 4. $\frac{3}{20}(1+2x)^{\frac{5}{3}}-\frac{3}{8}(1+2x)^{\frac{2}{3}}$.

5. $-\sqrt{(a^2-x^2)}$. 6. $-\log(c-x)$. 7. $-\frac{1}{6}\log(1-2x^3)$.

8. $\frac{1}{2}\log(x^2+a^2)+\tan^{-1}\left(\frac{x}{a}\right)$. 9. $-\frac{1}{b}\log(a+b\cos x)$. 10. $-\frac{1}{b(a+bx)}$.

11. $\frac{1}{9}\left[\frac{1}{1-3x}+\log(1-3x)\right]$. 12. $\log\left(1+\frac{1}{x}\right)-\frac{1}{x}$.

13. $\frac{2}{27}(2+3x)^{\frac{3}{2}}-\frac{4}{9}(2+3x)^{\frac{1}{2}}$. 14. $\frac{2}{3}(1+\log x)^{\frac{3}{2}}$. 15. $\frac{\log x}{\log a}$.

16. $\frac{x}{8}-\frac{\sin 4x}{32}$. 17. $\log(\sin x)$. 18. $\frac{1}{3}\log\left(\tan\frac{3x}{2}\right)$. 19. $-\frac{1}{3}\cot 3x$.

20. $\frac{1}{2}(x^2-8x-20)+12\log(x+2)+\frac{8}{x+2}$. 21. $\frac{\sin(ax-bx)}{2(a-b)}-\frac{\sin(ax+bx)}{2(a+b)}$.

22. $\frac{5x^2}{2}+15x-6\log(x-1)+41\log(x-2)$. 23. $\frac{3}{4}\log(x+3)+\frac{1}{4}\log(x-1)$.

24. $\frac{1}{4}\log\frac{1+x^2}{(1+x)^2}+\frac{1}{2}\tan^{-1}x$. 25. $\frac{1}{2}x-\frac{1}{12}\sin 6x$. 26. $\frac{x^2}{4}(2\log x-1)$.

27. $x\cos^{-1}x-\sqrt{(1-x^2)}$. 28. $\log\left(\tan\frac{\pi+2x}{4}\right)$. 29. $\frac{1}{8}\sin^4 2x$.

30. $\sqrt{(x^2-a^2)}+a\sin^{-1}\left(\frac{a}{x}\right)$. 31. $\frac{3}{32}(3-2x)^{\frac{8}{3}}-\frac{9}{20}(3-2x)^{\frac{5}{3}}$.

32. $\frac{1}{5}x^{\frac{5}{2}} - 5x^{\frac{1}{2}}$. 33. $\log x[\log(a\sqrt{x})]$. 34. $\log(1+\sin x)$.
35. $e^x(x^2-2x+2)$. 36. $-\cos(\log x)$. 37. $(\log x)^2$. 38. $\log(1+\log x)$.
39. $\frac{1}{2}(\tan^{-1}x)^2$. 40. $2\sqrt{(e^x+4)} - 2x + 4\log[\sqrt{(e^x+4)}-2]$.
41. $\dfrac{1}{a}\tan(ax+b) - x$. 42. $\dfrac{7^x}{\log 7}$. 43. $x\log_2\left(\dfrac{x}{e}\right)$. 44. $\sin^{-1}x + \sqrt{(1-x^2)}$.
45. $\frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}}$. 46. $\frac{1}{4}\tan^4 x$. 47. $-\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)$.
48. $\frac{1}{2}\tan^2 x + \log\cos x$. 49. $\frac{1}{2}\tan^{-1}(2\tan x)$.
50. $\dfrac{1}{a}\{\log x - \log[a+\sqrt{(a^2-x^2)}]\}$. 51. $\dfrac{\pi}{4} - \frac{2}{3}$. 52. $\frac{1}{2}$. 53. $\frac{1}{60}$. 54. $\frac{16}{35}$.
55. $\dfrac{\pi}{2}$. 56. $\dfrac{(4-\pi)\sqrt{2}}{8}$. 57. $\frac{2}{27}$. 58. 1. 59. $\dfrac{35\pi}{256}$. 60. 1.
61. $\log\frac{32}{27}$. 62. $-\frac{1}{4}+\frac{1}{2}\log 2$. 63. $\dfrac{\pi}{8}$. 67. $\dfrac{\pi}{8}$. 68. $\frac{8}{15}$.

EXAMPLES XV a (p. 237)

1. $\phi=\theta,\ p=a\sin^2\theta$; $\phi=\pi-\theta,\ p=a$; $\phi=\dfrac{3\pi}{4}-\dfrac{\theta}{2},\ p=\dfrac{a}{2}\sec\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$; $\phi=\alpha,\ p=r\sin\alpha$.

3. $p^2=ar$. 4. $r^{n+1}=a^n p$. 5. $\theta=\dfrac{\pi}{12},\ r=\dfrac{a}{2}\sqrt[4]{12}$. 6. $r=a$.

EXAMPLES XV b (p. 239)

1. $\frac{1}{4}\pi a^2$. 2. $\dfrac{3\pi a^2}{2}$. 3. $\dfrac{\pi}{4}<\theta<\dfrac{3\pi}{4}$. 4. $\frac{1}{2}a^2$. 5. $\frac{1}{2}c^2\log 3$.

6. Sum of areas of loops $=3\pi$; $r=2\cos\theta-1$ for $-\dfrac{2\pi}{3}<\theta<\dfrac{2\pi}{3}$; $\theta=\dfrac{2\pi}{3}$ to $\theta=\dfrac{4\pi}{3}$ for $r=1+2\cos\theta$. 8. $\frac{1}{2}ab\tan^{-1}\left(\dfrac{a\tan\alpha}{b}\right)$.

EXAMPLES XV c (p. 241)

1. $\dfrac{2r}{\pi}$ from centre. 2. $\dfrac{4r}{3\pi}$ from centre. 5. $\dfrac{\pi a}{4\sqrt{2}}$. 6. $\dfrac{4a}{5}$ from origin.

EXAMPLES XV d (p. 244)

1. $\frac{1}{27}(13\sqrt{13}-8)$. 2. $4a$. 3. $8a$. 4. $\frac{61}{27}$. 5. $6a$.
6. $\dfrac{c}{2}\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right)$. 8. $h+\frac{2}{3}a^2h^3$. 9. $a\sec\gamma\,(e^{\beta\cot\gamma} - e^{\alpha\cot\gamma})$.

EXAMPLES XV e (p. 248)

1. $\dfrac{13\sqrt{13}}{6}$. 2. $\dfrac{1}{2a}$; $\dfrac{1}{2a}$. 3. $\dfrac{(x^2+y^2)^{\frac{3}{2}}}{2c^2}$; $\dfrac{y^2}{c}$. 4. $r\operatorname{cosec}\alpha$; $\dfrac{4a}{3}\cos\dfrac{\theta}{2}$.
6. $4a\cos\dfrac{\theta}{2}$. 7. $3a\sin t\cos t$. 12. $\dfrac{1}{ab}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}$.

EXAMPLES XV f (p. 252)

1. $\pi r l$. 2. $\frac{8\pi a^2}{3}\left[\left(1+\frac{x}{a}\right)^{\frac{3}{2}}-1\right]$. 3. $\frac{1225\pi}{4}$ sq. in.

4. $\frac{12\pi a^2}{5}$. 5. $\frac{32\pi a^2}{5}$. 6. $2\pi a^2(2-\sqrt{2})$.

7. $\frac{\pi c^2}{4}\left(e^{\frac{2a}{c}}-e^{-\frac{2a}{c}}+\frac{4a}{c}\right)$; $\pi c\left[a\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)-c\left(e^{\frac{a}{c}}+e^{-\frac{a}{c}}\right)+2c\right]$.

8. Area of surface formed by revolution about $y=x\tan\alpha$. 9. $\frac{5\pi^2}{4}$ sq. in.

10. $\frac{1}{3}\pi r^2 h$; $\pi r\sqrt{(r^2+h^2)}$. 11. $\frac{2r}{\pi}$ from centre; $\pi(\pi+1)$ sq. in.

12. $\frac{r\sin\alpha}{\alpha}$ from centre; $4\pi r^2(\sin\alpha-\alpha\cos\alpha)$. 13. $\frac{4r}{3\pi}$ from centre.

14. 1 ft. 15. $16\pi(\pi-1)$ sq. in.

EXAMPLES XV g (p. 256)

1. In Fig. 176, ends of chain are on OX.

11. $\sqrt{(4ag)}$; $p=4a$, $n=\sqrt{\left(\frac{g}{4a}\right)}$, $\epsilon=0$; $\pi\sqrt{\left(\frac{a}{g}\right)}$. 12. $2m\cos\psi$.

REVISION PAPERS R. 18–24 (p. 257)

R. 18. 1. $-3\sin(6x+4)$; $\frac{2}{\sqrt{(7-12x-4x^2)}}$. 3. $-\frac{x}{y}$; $-\sqrt{\left(\frac{y}{x}\right)}$; $-\frac{y}{x}$.

4. $a=\frac{RE}{R^2+L^2p^2}$, $b=\frac{LpE}{R^2+L^2p^2}$. 5. $22 \cdot 4°$.

R. 19. 1. $-\frac{a}{x^2+a^2}$; $\frac{1}{2}\operatorname{cosec}^{\frac{1}{2}}(1-x)\cot(1-x)$; $\frac{5\pi a^6}{32}$.

2. $x\sec\phi+y\operatorname{cosec}\phi=a$; $\frac{3a}{2}\sin 2\phi$. 4. $\frac{b\delta b-a\delta a}{c}$. 5. $\left(\frac{\pi^2+4}{8\pi}, 0\right)$.

R. 20. 1. $\frac{1}{x-1}$; $-4\operatorname{cosec} 4x$; $\frac{1}{2}\log\frac{5}{3}$; 1. 2. $\frac{dV}{dt}=-kx$ where $V=\frac{1}{3}\pi x^3\tan^2\alpha$.

3. 315 ft. 4. 0·1823. 5. $\frac{12400}{\sqrt{3}}$ lb. $=3 \cdot 20$ tons.

R. 21. 1. $-\frac{1}{2}x^{-\frac{3}{2}}dx$; $ae^{ax}dx$; $\frac{b\,dx}{bx-a}$; $\sec^2 x\,dx$.

3. $\frac{2}{e}$. 5. $\frac{v_0}{k}(1-e^{-kt})$ ft.; 3·92 sec.

R. 22. 1. $\frac{4}{5}x^5-\frac{20}{3}x^3+25x+c$; $\log(3x^2-5x+7)+c$; $\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+c$;

$\frac{1}{16}(\sin 8\theta+4\sin 2\theta)+c$.

2. £122·1. 3. 0·7 per cent. 4. 164 ft.; 1·41 sec.

R. 23. **1.** $\frac{\pi}{2}$; $\log 3 - 1 = \cdot 0896$. **2.** $(p-x)(q-x)$.

3. $2e^{-\frac{1}{2}t}(6\cos 3t - \sin 3t)$; $-e^{-\frac{\pi}{6}} \doteq -0\cdot 592$. **5.** 1·36 tons.

R. 24. **1.** $\dfrac{1}{k(b-a)}\log\dfrac{a(b-x)}{b(a-x)}$. **2.** 92·1 per cent.

3. $\frac{1}{20}\log(x-2)+\frac{1}{5}\log(x+3)-\frac{1}{4}\log(x+2)+c$; π.

4. $\dfrac{a^2}{2}\log\left(\tan\dfrac{5\pi}{12}\right)$. **5.** $\frac{1}{2}a^2$.

MISCELLANEOUS EXAMPLES M 24—31 (p. 260).

M. 24. **2.** $3b \geqslant a^2$. **3.** $\dfrac{16\pi a^3}{5}$. **4.** $\theta = 90°$, $r=1$, min.;

$\theta = 54° 44$ or $125° 16'$, $r = \frac{4}{9}\sqrt{6} = 1\cdot 63$, max. **5.** $\dfrac{13\pi}{32}$.

M. 25. **1.** $e - \dfrac{3}{e}$. **2.** $\frac{1}{2}\pi a^2 b$.

3. $x^{\frac{1}{x}-2}(1-\log x)$; $(-1)^{n-1}\lfloor n-1 \left(\dfrac{1}{(x+1)^n}+\dfrac{1}{(x+3)^n}\right)$.

4. max. 8, min. 0. **5.** $r^2\cos 2\theta = a^2$.

M. 26. **1.** $83\frac{1}{3}$ min. **2.** 260 lb. **4.** $\dfrac{1}{e}$. **5.** $29\frac{1}{6}$ cu. in.

M. 27. **2.** 4, 1. **3.** $a=\frac{1}{2}$, $b=2$; 0, $2+\sqrt{2}$. **4.** $-bn$, $-am'$; larger force; 7071 men. **5.** $A < 9B$; $A = \frac{3}{2}$, $B = \frac{1}{2}$; $\sqrt{2}$.

M. 28. **1.** $\frac{1}{3}$. **2.** $\dfrac{\pi s_1{}^3}{6a}$. **4.** $y = \dfrac{gx^2}{3200}$; $\dfrac{1600}{g}[2\sqrt{3}+\log(2+\sqrt{3})] = 239$ ft.

5. 8350 ft. lb.

M. 29. **2.** $\dfrac{4\pi^3 a^2}{3}$. **3.** 6·84 in above vertex; 8·4 lb. **4.** $\dfrac{a^2}{2}\left(e-\dfrac{1}{e}\right)$

M. 30. **1.** $5\cot\dfrac{\theta}{2}$; $-\dfrac{5}{2}\operatorname{cosec}^2\dfrac{\theta}{2}\,\delta\theta$; 6·6 ft. **3.** $b - 2a\cos\theta$;

$a(b-2a\cos\theta)\,\delta\theta$; $\frac{1}{2}\pi ab - 2a^2$. **4.** 330 lb. **5.** $2\pi r^2(\pi-2)$.

M. 31. **3.** $\dfrac{a(1+e^{-b\pi})}{1+b^2}$. **4.** $\frac{3}{2}\sqrt[3]{(ax^2)}$.

EXAMPLES XVI a (p. 270)

1. x^2+y^2; $2i$; $\frac{1}{2}(i\sqrt{3}-1)$; -1. **2.** $\pm 2i$; $-4\pm 3i$. **4.** $a=3$, $b=5$.

5. $r=5$, $\theta = 2n\pi + 0\cdot 927$ or $r=-5$, $\theta=(2n+1)\pi+0\cdot 927$. **6.** -1, $1\pm i$.

13. $\cos 5\theta + i\sin 5\theta$; $2\cos\dfrac{\theta}{2}$. **16.** 0, 1, 0. **20.** $\cosh n\theta - \sinh n\theta = e^{-n\theta}$.

22. $\sec\theta$; $\sin\theta$. **24.** $\pm 1\cdot 32$.

EXAMPLES XVI b (p. 273)

3. $\dfrac{1}{\sqrt{(a^2+x^2)}}$.   4. $\dfrac{a}{a^2-x^2}$.   6. $\dfrac{1}{n}\cosh(nx)$; $\dfrac{1}{n}\sinh(nx)$.

8. $\dfrac{1}{n}\tanh(nx)$; $-\dfrac{1}{n}\coth(nx)$.   9. $\frac{1}{4}\sinh 2\theta$.

11. $\frac{1}{3}\sinh^{-1}(3x)$; $\cosh^{-1}\left(\dfrac{x}{2}\right)$.   12. $\cosh^{-1}\left(\dfrac{x+a}{b}\right)$; $\cosh^{-1}\left(\dfrac{x+5}{3}\right)$.

13. $\frac{1}{2}x\sqrt{(1+x^2)}-\frac{1}{2}\sinh^{-1}x$.   14. $\frac{1}{2}\cosh(ax+bx)-\frac{1}{2}\cosh(ax-bx)$; $\dfrac{\sinh(ax+bx)}{2(a+b)}-\dfrac{\sinh(ax-bx)}{2(a-b)}$.   16. $s=c\sinh\left(\dfrac{x}{c}\right)$.

18. $c\cosh^2\left(\dfrac{x}{c}\right)$.   19. $x\cosh x-\sinh x$; $\frac{1}{4}e^{2x}-\frac{1}{2}x$.

20. $\log\left(\tanh\dfrac{x}{2}\right)$; $x\sinh^{-1}x-\sqrt{(1+x^2)}$.

EXAMPLES XVII a (p. 279)

10. $1+x-\dfrac{x^3}{3}-\dfrac{x^4}{6}-\dfrac{x^5}{30}$.   11. $\log 2+\dfrac{x}{2}+\dfrac{x^2}{8}$.   12. $1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$.

13. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4$.

EXAMPLES XVII b (p. 281)

1. $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots$.   2. $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\dots$.

3. $\log\dfrac{1+x}{1-x}=2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\dots\right)$.   4. $\dfrac{2}{(1-x)^3}$.   5. $(1+x)\log(1+x)-x$.

6. $(1-x^2)^{-\frac{3}{2}}$.   7. $\cos x-x\sin x$.   8. $\dfrac{\sinh x}{2x}+\frac{1}{2}\cosh x$.

9. $\frac{1}{2}\tan^{-1}x+\dfrac{x}{2(1+x^2)}$.   11. $\dfrac{1}{2}\left\{\dfrac{(2x)^2}{\lfloor 2}-\dfrac{(2x)^4}{\lfloor 4}+\dfrac{(2x)^6}{\lfloor 6}-\dots\right\}$.

12. $\dfrac{1}{2}\left\{2x-\dfrac{(2x)^3}{\lfloor 3}+\dfrac{(2x)^5}{\lfloor 5}-\dots\right\}$.   14. $x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dfrac{x^7}{7}+\dots$.

15. $x-\frac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}-\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\dots$.

EXAMPLES XVII d (p. 291)

1. $\frac{1}{9}$.   2. 1.   3. 2.   4. $\frac{4}{9}$.   5. $-\frac{1}{3}$.   6. 0.   7. $\frac{1}{2}$.   8. $\frac{1}{2}$.
9. 0.   10. $a_0=1$; $a_1=-2n^2$.   11. $a_0=2n$; $a_1=\frac{4}{3}n(1-n^2)$.

MISCELLANEOUS EXAMPLES M. 32—35 (p. 291)

M. 32. 1. $-\dfrac{a}{(\theta-a)^2}$; $\dfrac{x^2}{2}-\dfrac{x^4}{12}$.   3. $\dfrac{a}{3}$.   4. πa.   5. point of inflexion.

M. 33. 3. $y=a\,.\,e^{-\frac{x}{y}}$.   4. $4a^3\left(1-\dfrac{2}{\pi}\right)$; $\dfrac{a(\pi+2)}{16}$.

M. 34. 1. $\frac{4}{5}$ ft.   2. $\frac{1}{2}\pi^2-\frac{1}{4}\pi$ sq. in.   3. $(\sqrt[4]{\tfrac{5}{3}}, 3\tfrac{1}{3})$.

M. 35. 1. $\angle ACP$ obtuse, 50 ft. min.; $\angle ACP$ acute, 62·8 ft. min.
4. $\sqrt{\tfrac{128}{315}}$.   5. $\sqrt{\{\frac{1}{2}(a^2+p^2)\}}$.

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